The distribution of real polynomials with bounded roots

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On the distribution of polynomials with bounded roots,
I. Polynomials with real coefficients

Akiyama, S. and Pethő, A.

Journal of Mathematical Society of Japan, to appear

http://www.inf.unideb.hu/~pethoe/Publications.html

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Main interest: $v_d^{(s)}$ and $\frac{v_d^{(s)}}{v_d^{(0)}}$

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$$\mathbf{x} = (x_1, \ldots, x_d) \to (x_2, \ldots, x_d, -\lfloor \mathbf{rx} \rfloor)$$

is called the d - dimensional SRS associated with $\mathbf{r}$ (AKIYAMA et al. 2005)

where $\mathbf{rx} = \sum_{i=1}^{d} r_i x_i$ denotes the scalar product of $\mathbf{r}$ and $\mathbf{x}$ and $\lfloor y \rfloor$ the largest integer less than or equal to some real y. (floor)

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$\mathcal{D}_d^{(0)} \subseteq \mathcal{D}_d$

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Basic facts and previous results

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- Study of boundary of $\mathcal{E}_d$: KIRSCHENHOFER, PETHŐ, SURER, THUSWALDNER 2010
- $v_d = \begin{cases} 2^{2m^2} \prod_{j=1}^{m} \frac{(j-1)!^4}{(2j-1)!^2} & \text{if } d = 2m \\ 2^{2m^2+2m+1} \prod_{j=1}^{m} \frac{j!^2 (j-1)!^2}{(2j-1)!(2j+1)!} & \text{if } d = 2m+1 \end{cases}$ (FAM 1989)

Theorem: $v_d^{(s)} = \frac{1}{r!s!} \int_{D_{r,s}} |\Delta_r| \, \Delta_s \Delta_{r,s} dX$

where $D_{r,s} = [-1,1]^r \times \prod_{i=1}^{s} \left([-2\sqrt{z_i}, 2\sqrt{z_i}] \times [0,1]\right)$

$$dX = dx_1 \dots dx_r dy_1 dz_1 \dots dy_s dz_s$$

$$\Delta_r = \prod_{j=1}^{r} \prod_{k=j+1}^{r} (x_j - x_k)$$

$$\Delta_s = \prod_{j=1}^{s} \prod_{k=j+1}^{s} \mathrm{Res}_X(R_j(X), R_k(X))$$

$$\Delta_{r,s} = \prod_{j=1}^{r} \prod_{k=1}^{s} R_k(x_j)$$

$$R_j(X) = X^2 - y_j X + z_j$$

$\mathrm{Res}_X(P(x), Q(x))$ ... Resultant of P and Q

$$\mathrm{Res}_X(R_j(X), R_k(X)) = -y_j y_k (z_j + z_k) + y_j^2 z_k + y_k^2 z_j + (z_j - z_k)^2$$

Idea of proof: $v_d^{(s)} = \lambda_d(\mathcal{E}_d^{(s)}) = \int_{\mathcal{E}_d^{(s)}} dp_0 \dots dp_{d-1}$

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- For a polynomial $P(X) = X^d + p_{d-1}X^{d-1} + \dots + p_0 \in \mathbb{R}[X]$ with roots $(x_1, \dots, x_d) \in \mathbb{R}^r \times \mathbb{C}^{2s}$ use Vieta's formulas to express its coefficients in terms of its roots and define a substitution.

  $p_j = (-1)^{d-j} S_{d-j}$ where $S_j(x_1, \dots, x_d) = \sum_{1 \le i_1 < \dots < i_j \le d} x_{i_1} \dots x_{i_j}$

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- $p_0, \dots, p_{d-1} \to x_1, \dots, x_d \to y_1, \dots, y_d$

  by $y_j = x_j$ for $j \in \{1, \dots, r\}$

  $y_{r+2j-1} = x_{r+2j-1} + x_{r+2j}$, $y_{r+2j} = x_{r+2j-1}x_{r+2j}$ for $j \in \{1, \dots, s\}$

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- New boundaries: Union of unit intervals and unit disks.
- Compute determinant of Jacobian matrix $J = \left(\frac{\partial S_i(x_1, \dots, x_d)}{\partial y_j} \right)_{1 \le i,j \le d}$

$$\det(J) = \prod_{j=1}^r \prod_{k=j+1}^r (y_j - y_k) \prod_{j=1}^r \prod_{k=1}^s R_k(y_j) \times \prod_{j=1}^s \prod_{k=j+1}^s \mathrm{Res}_X(R_j(X), R_k(X))$$

Values of $v_d^{(s)}$:

d	$v_d^{(0)}$	$v_d^{(1)}$	$v_d^{(2)}$
2	$\frac{4}{3}$	$\frac{8}{3}$	
3	$\frac{16}{45}$	$\frac{224}{45}$	
4	$\frac{64}{1575}$	$\frac{1664}{525}$	$\frac{2048}{525}$
5	$\frac{1024}{496125}$	$\frac{428032}{496125}$	$\frac{3334144}{496125}$
6	$\frac{16384}{343814625}$	$\frac{1114112}{10418625}$	$\frac{93519872}{22920975}$
7	$\frac{524288}{1032475318875}$	$\frac{2124414976}{344158439625}$	$\frac{379792130048}{344158439625}$
8	$\frac{16777216}{6643978676960625}$	$\frac{1114476904448}{6643978676960625}$	$\frac{313947815149568}{221465955898687 5}$
9	$\frac{4294967296}{72681804736610757 1875}$	$\frac{92376156602368}{42754002786241621875}$	$\frac{12626155878219776}{14335661683749 65625}$

Values of $v_d/v_d^{(0)}$ and $v_d^{(s)}/v_d^{(0)}$:

d	$v_d/v_d^{(0)}$	$v_d^{(1)}/v_d^{(0)}$	$v_d^{(2)}/v_d^{(0)}$	$v_d^{(3)}/v_d^{(0)}$
2	3	2		
3	15	14		
4	175	78	96	
5	3675	418	3256	
6	169785	2244	85620	81920
7	14567553	12156	2173188	12382208
8	2678348673	66428	56138244	1447738880
9	930152232009	365636	1490456292	164885467424

Theorem: $v_d^{(s)} \in \mathbb{Q}$ and $v_d / v_d^{(0)} \in \mathbb{Z}$

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Hope for general result!

Theorem: $v_d^{(0)} = \frac{2^{d(d+1)/2}}{d!} S_d(1,1,1/2)$

$$\text{where } S_n(\alpha,\beta,\gamma) = \int_{[0,1]^n} \prod_{j=1}^{n} t_j^{\alpha-1}(1-t_j)^{\beta-1} \times \left| \prod_{j=1}^{n} \prod k = j+1^n (t_j - t_k) \right|^{2\gamma} dt_1 \ldots dt_n = \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma((j+1)\gamma+1)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(\gamma+1)}$$

$$S_d(1,1,1/2) = \frac{1}{\prod_{i=0}^{d-1} \binom{2i+1}{i}}$$

(SELBERG 1944)

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$\left|\prod_{j=1}^n \prod k = j+1^n (t_j - t_k)\right|^{2\gamma} dt_1 \ldots dt_n$
$= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma((j+1)\gamma+1)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(\gamma+1)}$

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Theorem: $v_d^{(1)} = 2^{(d-1)(d-2)/2-2} \times$
$\sum_{j=0}^{d-2}\sum_{k=0}^{d-2-j}\left(\frac{(-1)^{d-k}2^{2d-2-2k-j}}{j!k!(d-2-j-k)!} \times\right.$
$B_{d-2}(d-2-k, d-2-k-j) \times$
$\left.\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz\right)$

where $B_d(j,k) = \prod_{i=1}^{k} \frac{2+(d-i-1)/2}{3+(2d-i-1)/2} \times$
$\frac{\prod_{i=1}^{j}(1+(d-i)/2)\prod_{i=1}^{k}(1+(d-i)/2)}{\prod_{i=1}^{j+k}(2+(2d-i-1)/2)} \times$
$S_d(1,1,1/2)$

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \Bigg(2^{(d-1)(d-2)/2-2}$$

$$\sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \Bigg(\frac{(-1)^{d+k} 2^{2d-2-2k-j}}{j!k!(d-2-j-k)!} \prod_{i=1}^{d-2-k-j} \frac{2+(d-2-i-1)/2}{3+(2(d-2)-i-1)/2}$$

$$\frac{\prod_{i=1}^{d-2-k}(1+(d-2-i)/2) \prod_{i=1}^{d-2-k-j}(1+(d-2-i)/2)}{\prod_{i=1}^{d-2-k+d-2-k-j}(2+(2(d-2)-i-1)/2)}$$

$$\frac{1}{\prod_{i=0}^{d-2-1} \binom{2i+1}{i}} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz \Bigg) \Bigg) /$$

$$\left(\frac{2^{d(d+1)/2}}{d!} \frac{1}{\prod_{i=0}^{d-1} \binom{2i+1}{i}} \right)$$

$$
\begin{aligned}
\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} \Bigg(& (-1)^{d+k} \frac{1}{2^{j+2k+3}} \frac{d!}{j!k!(d-j-k-2)!} \\
& \prod_{i=1}^{d-j-k-2} \frac{d-i+1}{2d-i+1} \frac{\prod_{i=1}^{d-k-2}(d-i) \prod_{i=1}^{d-j-k-2}(d-i)}{\prod_{i=1}^{2d-j-2k-4}(2d-i-1)} \frac{\prod_{i=0}^{d-1} \binom{2i+1}{i}}{\prod_{i=0}^{d-3} \binom{2i+1}{i}} \\
& \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz \Bigg)
\end{aligned}
$$

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} \Bigg((-1)^{d+k} \frac{1}{2^{j+2k+3}} \frac{d!}{j!k!(d-j-k-2)!} \frac{\frac{d!}{(j+k+2)!}}{\frac{(2d)!}{(d+j+k+2)!}} \frac{\frac{(d-1)!}{(k+1)!} \frac{(d-1)!}{(j+k+1)!}}{\frac{(2d-2)!}{(j+2k+2)!}} \frac{(2d-3)!}{(d-2)!(d-1)!} \frac{(2d-1)!}{(d-1)!d!} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz \Bigg)$$

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} \Bigg((-1)^{d+k} \frac{1}{2^{j+2k+5}} \frac{(d+j+k+2)!(j+2k+2)!}{(d-j-k-2)!j!(j+k+2)!(j+k+1)!(k+1)!k!} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz \Bigg)$$

Computation of $\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz$

Computation of $\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz$

Initial idea: Binomial theorem

$$\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y + z + 1)^k \, dy \, dz =$$

$$\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz =$$

$$= \sum_{r=0}^{k} \binom{k}{r} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^{r+j} (z+1)^{k-r} \, dy \, dz$$

$$\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz =$$

$$= \sum_{r=0}^{k} \binom{k}{r} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^{r+j} (z+1)^{k-r} \, dy \, dz$$

$$= \sum_{r=0}^{k} \binom{k}{r} \frac{2^{j+r+1}(1+(-1)^{j+r})}{j+r+1} \int_0^1 (z+1)^{k-r} z^{(j+r+1)/2} \, dz$$

$$
\begin{aligned}
\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz &= \\
&= \sum_{r=0}^{k} \binom{k}{r} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^{r+j} (z+1)^{k-r} \, dy \, dz \\
&= \sum_{r=0}^{k} \binom{k}{r} \frac{2^{j+r+1}(1+(-1)^{j+r})}{j+r+1} \int_0^1 (z+1)^{k-r} z^{(j+r+1)/2} \, dz \\
&= \sum_{r=0}^{k} \binom{k}{r} \frac{2^{j+r+1}(1+(-1)^{j+r})}{j+r+1} \sum_{s=0}^{k-r} \binom{k-r}{s} \int_0^1 z^{(j+r+1)/2+s} \, dz
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz = \\
&\quad = \sum_{r=0}^{k} \binom{k}{r} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^{r+j} (z+1)^{k-r} \, dy \, dz \\
&\quad = \sum_{r=0}^{k} \binom{k}{r} \frac{2^{j+r+1}(1+(-1)^{j+r})}{j+r+1} \int_0^1 (z+1)^{k-r} z^{(j+r+1)/2} \, dz \\
&\quad = \sum_{r=0}^{k} \binom{k}{r} \frac{2^{j+r+1}(1+(-1)^{j+r})}{j+r+1} \sum_{s=0}^{k-r} \binom{k-r}{s} \int_0^1 z^{(j+r+1)/2+s} \, dz \\
&\quad = \sum_{r=0}^{k} \binom{k}{r} \frac{2^{j+r+1}(1+(-1)^{j+r})}{j+r+1} \sum_{s=0}^{k-r} \binom{k-r}{s} \frac{2}{j+r+2s+3}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy\, dz &= \\
&= \sum_{r=0}^{k} \binom{k}{r} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^{r+j} (z+1)^{k-r} \, dy\, dz \\
&= \sum_{r=0}^{k} \binom{k}{r} \frac{2^{j+r+1}(1+(-1)^{j+r})}{j+r+1} \int_0^1 (z+1)^{k-r} z^{(j+r+1)/2} \, dz \\
&= \sum_{r=0}^{k} \binom{k}{r} \frac{2^{j+r+1}(1+(-1)^{j+r})}{j+r+1} \sum_{s=0}^{k-r} \binom{k-r}{s} \int_0^1 z^{(j+r+1)/2+s} \, dz \\
&= \sum_{r=0}^{k} \binom{k}{r} \frac{2^{j+r+1}(1+(-1)^{j+r})}{j+r+1} \sum_{s=0}^{k-r} \binom{k-r}{s} \frac{2}{j+r+2s+3} \\
&= \sum_{r=0}^{k} \sum_{s=0}^{k-r} \frac{2^{j+r+2}(1+(-1)^{j+r})k!}{(j+r+1)(j+r+2s+3)(k-r-s)!r!s!}
\end{aligned}
$$

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} \Bigg((-1)^{d+k} \frac{1}{2^{j+2k+5}} \frac{(d+j+k+2)!(j+2k+2)!}{(d-j-k-2)!j!(j+k+2)!(j+k+1)!(k+1)!k!} \sum_{r=0}^{k} \sum_{s=0}^{k-r} \frac{2^{j+r+2}(1+(-1)^{j+r})k!}{(j+r+1)(j+r+2s+3)(k-r-s)!r!s!} \Bigg)$$

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$$= \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} \Bigg((-1)^{d-k} \sum_{r=0}^{k} \sum_{s=0}^{k-r} \Bigg(\frac{1+(-1)^{j+r}}{2} \frac{1}{2^{2k-r+2}} \frac{j+k+2}{j+r+2s+3} \binom{d+j+k+2}{d} \binom{d}{j+k+2} \binom{j+2k+2}{k+1} \binom{j+k+1}{k-r} \binom{k-r}{s} \binom{j+r}{j} \Bigg) \Bigg)$$

$$f(d,j,k,r,s) \;:=\; \frac{j+k+2}{j+r+2s+3}\binom{d+j+k+2}{d}\binom{d}{j+k+2}\binom{j+k+1}{k-r}\binom{k-r}{s}$$

$$f(d,j,k,r,s) \;:=\; \frac{j+k+2}{j+r+2s+3}\binom{d+j+k+2}{d}\binom{d}{j+k+2}\binom{j+k+1}{k-r}\binom{k-r}{s}$$

Show: $\nu_p(f(d,j,k,r,s)) \geq 0$ for all $p \in \mathbb{P}$

$$f(d,j,k,r,s) := \frac{j+k+2}{j+r+2s+3}\binom{d+j+k+2}{d}\binom{d}{j+k+2}\binom{j+k+1}{k-r}\binom{k-r}{s}$$

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Implies that $2^n \frac{v_d^{(1)}}{v_d^{(0)}} \in \mathbb{Z}$ for some $n \in \mathbb{N}_0$

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Use Legendre's theorem: $\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$ for all $n \in \mathbb{N}$ and $p \in \mathbb{P}$

$$
\begin{aligned}
\nu_p(f(d,j,k,r,s)) = \sum_{i=1}^{\infty} \Bigg(& \left\lfloor \frac{j+r+2s+2}{p^i} \right\rfloor - \left\lfloor \frac{j+r+2s+3}{p^i} \right\rfloor + \\
& \left\lfloor \frac{j+k+2}{p^i} \right\rfloor - \left\lfloor \frac{j+k+1}{p^i} \right\rfloor + \\
& \left\lfloor \frac{d+j+k+2}{p^i} \right\rfloor - \left\lfloor \frac{d}{p^i} \right\rfloor - \left\lfloor \frac{j+k+2}{p^i} \right\rfloor + \\
& \left\lfloor \frac{d}{p^i} \right\rfloor - \left\lfloor \frac{j+k+2}{p^i} \right\rfloor - \left\lfloor \frac{d-j-k-2}{p^i} \right\rfloor + \\
& \left\lfloor \frac{j+k+1}{p^i} \right\rfloor - \left\lfloor \frac{k-r}{p^i} \right\rfloor - \left\lfloor \frac{j+r+1}{p^i} \right\rfloor + \\
& \left\lfloor \frac{k-r}{p^i} \right\rfloor - \left\lfloor \frac{s}{p^i} \right\rfloor - \left\lfloor \frac{k-r-s}{p^i} \right\rfloor \Bigg)
\end{aligned}
$$

For $q \in \mathbb{N}$ define

$$
\begin{aligned}
n_1(d,j,k,r,s,q) &:= \left\lfloor \frac{j+r+2s+2}{q} \right\rfloor - \left\lfloor \frac{j+r+2s+3}{q} \right\rfloor \\
n_2(d,j,k,r,s,q) &:= \left\lfloor \frac{j+k+2}{q} \right\rfloor - \left\lfloor \frac{j+k+1}{q} \right\rfloor \\
n_3(d,j,k,r,s,q) &:= \left\lfloor \frac{d+j+k+2}{q} \right\rfloor - \left\lfloor \frac{d}{q} \right\rfloor - \left\lfloor \frac{j+k+2}{q} \right\rfloor \\
n_4(d,j,k,r,s,q) &:= \left\lfloor \frac{d}{q} \right\rfloor - \left\lfloor \frac{j+k+2}{q} \right\rfloor - \left\lfloor \frac{d-j-k-2}{q} \right\rfloor \\
n_5(d,j,k,r,s,q) &:= \left\lfloor \frac{j+k+1}{q} \right\rfloor - \left\lfloor \frac{k-r}{q} \right\rfloor - \left\lfloor \frac{j+r+1}{q} \right\rfloor \\
n_6(d,j,k,r,s,q) &:= \left\lfloor \frac{k-r}{q} \right\rfloor - \left\lfloor \frac{s}{q} \right\rfloor - \left\lfloor \frac{k-r-s}{q} \right\rfloor
\end{aligned}
$$

We define $[a]_b := a \bmod b$ (remainder, not residue class!) and use the identity $\left\lfloor \frac{a}{b} \right\rfloor = \frac{a-[a]_b}{b}$ to rearrange all mappings to involve only modulo functions modulo q:

$$n_1(d,j,k,r,s,q) = \frac{-[j+r+2s+2]_q + [j+r+2s+3]_q - 1}{q}$$

$$n_2(d,j,k,r,s,q) = \frac{-[j+k+2]_q + [j+k+1]_q + 1}{q}$$

$$n_3(d,j,k,r,s,q) = \frac{-[d+j+k+2]_q + [d]_q + [j+k+2]_q}{q}$$

$$n_4(d,j,k,r,s,q) = \frac{-[d]_q + [j+k+2]_q + [d-j-k-2]_q}{q}$$

$$n_5(d,j,k,r,s,q) = \frac{-[j+k+1]_q + [k-r]_q + [j+r+1]_q}{q}$$

$$n_6(d,j,k,r,s,q) = \frac{-[k-r]_q + [s]_q + [k-r-s]_q}{q}$$

Then it is clear that
$n_1(d, j, k, r, s, q) \in \{-1, 0\}$ and
$n_i(d, j, k, r, s, q) \in \{0, 1\}$ for $i \in \{2, 3, 4, 5, 6\}$

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Therefore one can assume w.l.o.g. that
$0 \leq d < q$, $0 \leq j < q$, $0 \leq k < q$, $0 \leq r < q$, $0 \leq s < q$.

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Furthermore it is clear that the result of any of the mappings will be invariant to addition of multiples of q to some or all arguments.

Therefore one can assume w.l.o.g. that
$0 \le d < q$, $0 \le j < q$, $0 \le k < q$, $0 \le r < q$, $0 \le s < q$.

If we define $u := \frac{d}{q}$, $v := \frac{j}{q}$, $w := \frac{k}{q}$, $x := \frac{r}{q}$, $y := \frac{s}{q}$, $z := \frac{1}{q}$ we get
$0 \le u < 1$, $0 \le v < 1$, $0 \le w < 1$, $0 \le x < 1$,$0 \le y < 1$ and $0 < z \le 1$
and:

$$n_1(d,j,k,r,s,q) = \lfloor v+x+2y+2z \rfloor - \lfloor v+x+2y+3z \rfloor$$
$$n_2(d,j,k,r,s,q) = \lfloor v+w+2z \rfloor - \lfloor v+w+z \rfloor$$
$$n_3(d,j,k,r,s,q) = \lfloor u+v+w+2z \rfloor - \lfloor u \rfloor - \lfloor v+w+2z \rfloor$$
$$n_4(d,j,k,r,s,q) = \lfloor u \rfloor - \lfloor v+w+2z \rfloor - \lfloor u-v-w-2z \rfloor$$
$$n_5(d,j,k,r,s,q) = \lfloor v+w+z \rfloor - \lfloor w-x \rfloor - \lfloor v+x+z \rfloor$$
$$n_6(d,j,k,r,s,q) = \lfloor w-x \rfloor - \lfloor y \rfloor - \lfloor w-x-y \rfloor$$

It is possible to give explicit representations of these piecewise constant functions under the given constraints for u, v, w, x, y, and z.

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Using this explicit representations one can verify (by computer assistance) that $n_1(d, j, k, r, s, q)$ can not be -1 when all other functions are 0.

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This proves $2^n \frac{v_d^{(1)}}{v_d^{(0)}} \in \mathbb{Z}$ for some $n \in \mathbb{N}_0$

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This proves $2^n \frac{v_d^{(1)}}{v_d^{(0)}} \in \mathbb{Z}$ for some $n \in \mathbb{N}_0$

Similar idea is used in main reference to show that $\frac{v_d}{v_d^{(0)}} \in \mathbb{Z}$

Computation of $\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y + z + 1)^k \, dy \, dz$

Another try: Repeated integration in parts

$$\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz =$$

$$\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz =$$

$$= \int_{-2}^{2} \int_{\frac{y^2}{4}}^{1} y^j (y+z+1)^k \, dz \, dy$$

$$\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz =$$

$$= \int_{-2}^{2} \int_{\frac{y^2}{4}}^{1} y^j (y+z+1)^k \, dz \, dy$$

$$= \frac{1}{k+1} \left(\int_{-2}^{2} y^j (y+2)^{k+1} \, dy - \int_{-2}^{2} y^j (y/2+1)^{2k+2} \, dy \right)$$

$$\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz =$$

$$= \int_{-2}^{2} \int_{\frac{y^2}{4}}^{1} y^j (y+z+1)^k \, dz \, dy$$

$$= \frac{1}{k+1} \left(\int_{-2}^{2} y^j (y+2)^{k+1} \, dy - \int_{-2}^{2} y^j (y/2+1)^{2k+2} \, dy \right) \quad (y/2 \to y)$$

$$= \frac{1}{k+1} \left(2^{j+k+2} \int_{-1}^{1} y^j (y+1)^{k+1} \, dy - 2^{j+1} \int_{-1}^{1} y^j (y+1)^{2k+2} \, dy \right)$$

$$\int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y + z + 1)^k \, dy \, dz =$$

$$= \int_{-2}^{2} \int_{\frac{y^2}{4}}^{1} y^j (y + z + 1)^k \, dz \, dy$$

$$= \frac{1}{k+1} \left(\int_{-2}^{2} y^j (y+2)^{k+1} \, dy - \int_{-2}^{2} y^j (y/2 + 1)^{2k+2} \, dy \right) \quad (y/2 \to y)$$

$$= \frac{1}{k+1} \left(2^{j+k+2} \int_{-1}^{1} y^j (y+1)^{k+1} \, dy - 2^{j+1} \int_{-1}^{1} y^j (y+1)^{2k+2} \, dy \right)$$

$$= \frac{2^{j+2k+4}}{k+1} \left(\sum_{r=1}^{j+1} \frac{(-2)^{r-1} (j)_{r-1}}{(k+r+1)_r} - \sum_{r=1}^{j+1} \frac{(-2)^{r-1} (j)_{r-1}}{(2k+r+2)_r} \right)$$

where $(x)_j := \prod_{i=0}^{j-1} (x - i)$.

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} \Bigg((-1)^{d+k} \frac{1}{2^{j+2k+5}} \frac{(d+j+k+2)!(j+2k+2)!}{(d-j-k-2)!j!(j+k+2)!(j+k+1)!(k+1)!k!} \frac{2^{j+2k+4}}{k+1} \left(\sum_{r=1}^{j+1} \frac{(-2)^{r-1}(j)_{r-1}}{(k+r+1)_r} - \sum_{r=1}^{j+1} \frac{(-2)^{r-1}(j)_{r-1}}{(2k+r+2)_r} \right) \Bigg)$$

$$
\begin{aligned}
\frac{v_d^{(1)}}{v_d^{(0)}} &= \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} \Bigg((-1)^{d+k} \frac{1}{2^{j+2k+5}} \\
&\quad \frac{(d+j+k+2)!(j+2k+2)!}{(d-j-k-2)!j!(j+k+2)!(j+k+1)!(k+1)!k!} \\
&\quad \frac{2^{j+2k+4}}{k+1} \left(\sum_{r=1}^{j+1} \frac{(-2)^{r-1}(j)_{r-1}}{(k+r+1)_r} - \sum_{r=1}^{j+1} \frac{(-2)^{r-1}(j)_{r-1}}{(2k+r+2)_r} \right) \Bigg) \\
&= \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} \Bigg((-1)^{d+k+1} \binom{d}{j+k+2} \binom{d+j+k+2}{d} \frac{j+k+2}{j+2k+3} \\
&\quad \Bigg(\sum_{r=1}^{j+1} (-2)^{r-2} \binom{j+2k+3}{2k+r+2} \binom{2k+r+2}{k+1} - \\
&\quad \sum_{r=1}^{j+1} (-2)^{r-2} \binom{j+2k+3}{2k+r+2} \binom{2k+2}{k+1} \Bigg) \Bigg)
\end{aligned}
$$

Applying the transformations $j+k+2 \to a$ and $k+1 \to b$ one gets

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{a=2}^{d}\sum_{b=0}^{a-1}\sum_{r=1}^{a-b}(-1)^{d+b}(-2)^{r-2}\frac{a}{a+b}\binom{d}{a}\binom{d+a}{d}\binom{a+b}{2b+r}\binom{2b+r}{b} - \sum_{a=2}^{d}\sum_{b=0}^{a-1}\sum_{r=1}^{a-b}(-1)^{d+b}(-2)^{r-2}\frac{a}{a+b}\binom{d}{a}\binom{d+a}{d}\binom{a+b}{2b+r}\binom{2b}{b}$$

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$$\sum_{a=2}^{d}\sum_{b=0}^{a-1}\sum_{r=1}^{a-b}(-1)^{d+b}(-2)^{r-2}\frac{a}{a+b}\binom{d}{a}\binom{d+a}{d}\binom{a+b}{2b+r}\binom{2b}{b}$$

$$= \sum_{a=2}^{d}\sum_{r=1}^{a}\sum_{b=0}^{a-r}(-1)^{d+b}(-2)^{r-2}\frac{a}{a+b}\binom{d}{a}\binom{d+a}{d}\binom{a+b}{2b+r}\binom{2b+r}{b} -$$

$$\sum_{a=2}^{d}\sum_{r=1}^{a}\sum_{b=0}^{a-r}(-1)^{d+b}(-2)^{r-2}\frac{a}{a+b}\binom{d}{a}\binom{d+a}{d}\binom{a+b}{2b+r}\binom{2b}{b}$$

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$$\sum_{a=2}^{d}\sum_{b=0}^{a-1}\sum_{r=1}^{a-b}(-1)^{d+b}(-2)^{r-2}\frac{a}{a+b}\binom{d}{a}\binom{d+a}{d}\binom{a+b}{2b+r}\binom{2b}{b}$$

$$= \sum_{a=2}^{d}\sum_{r=1}^{a}\sum_{b=0}^{a-r}(-1)^{d+b}(-2)^{r-2}\frac{a}{a+b}\binom{d}{a}\binom{d+a}{d}\binom{a+b}{2b+r}\binom{2b+r}{b} -$$

$$\sum_{a=2}^{d}\sum_{r=1}^{a}\sum_{b=0}^{a-r}(-1)^{d+b}(-2)^{r-2}\frac{a}{a+b}\binom{d}{a}\binom{d+a}{d}\binom{a+b}{2b+r}\binom{2b}{b}$$

$$= \sum_{a=2}^{d}(-1)^{d}a\binom{d}{a}\binom{d+a}{d}\sum_{r=1}^{a}(-2)^{r-2}$$

$$\left(\sum_{b=0}^{a-r}(-1)^{b}\frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b} - \sum_{b=0}^{a-r}(-1)^{b}\frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b}\right)$$

Simplification of $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b}$

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If $r = a$ then $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b} = \frac{1}{a}$

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If $r = a$ then $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b} = \frac{1}{a}$

If $r \neq a$ it follows from
$(-1)^k\binom{k-n-1}{k} = \binom{n}{k}$, $(n \in \mathbb{Z}, k \in \mathbb{N}_0)$
and from Vandermonde's identity
$\sum_{k=0}^{n}\binom{n}{k}\binom{s}{k+t} = \sum_{k=0}^{n}\binom{n}{k}\binom{s}{n+t-k} = \binom{n+s}{n+t}$, $(n \in \mathbb{N}_0, s \in \mathbb{Z}, t \in \mathbb{N}_0)$
that

$$\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b} = \frac{1}{a-r}\sum_{b=0}^{a-r}(-1)^b\binom{a-r}{b}\binom{a+b-1}{b+r}$$

Simplification of $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b}$

If $r = a$ then $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b} = \frac{1}{a}$

If $r \neq a$ it follows from
$(-1)^k\binom{k-n-1}{k} = \binom{n}{k}$, $(n \in \mathbb{Z}, k \in \mathbb{N}_0)$
and from Vandermonde's identity
$\sum_{k=0}^{n}\binom{n}{k}\binom{s}{k+t} = \sum_{k=0}^{n}\binom{n}{k}\binom{s}{n+t-k} = \binom{n+s}{n+t}$, $(n \in \mathbb{N}_0, s \in \mathbb{Z}, t \in \mathbb{N}_0)$
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$$\begin{aligned}\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b} &= \frac{1}{a-r}\sum_{b=0}^{a-r}(-1)^b\binom{a-r}{b}\binom{a+b-1}{b+r} \\ &= \frac{(-1)^r}{a-r}\sum_{b=0}^{a-r}\binom{a-r}{b}\binom{r-a}{b+r}\end{aligned}$$

Simplification of $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b}$

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Simplification of $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b}$

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Simplification of $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b}$

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that

$$\begin{aligned}
\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b} &= \frac{1}{a-r}\sum_{b=0}^{a-r}(-1)^b\binom{a-r}{b}\binom{a+b-1}{b+r} \\
&= \frac{(-1)^r}{a-r}\sum_{b=0}^{a-r}\binom{a-r}{b}\binom{r-a}{b+r} \\
&= \frac{(-1)^r}{a-r}\binom{0}{a} \\
&= 0
\end{aligned}$$

Therefore: $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b} = \frac{1}{a}\delta_{r,a}$

Simplification of $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b}$

Simplification of $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b}$

We will adapt an example given in the book Concrete Mathematics (GRAHAM, KNUTH, PATASHNIK 1994) and use the identities

$\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k}$, $(n \in \mathbb{N}_0, k \in \mathbb{N}_0)$

$\sum_{k=0}^{n}(-1)^k \frac{1}{k+m+1}\binom{n+k}{2k}\binom{2k}{k} = (-1)^n \frac{m!n!}{(m+n+1)!}\binom{m}{n}$, $(m \in \mathbb{N}_0, n \in \mathbb{N}_0)$

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$$\begin{aligned}
\sum_{b=0}^{a-r}(-1)^b\frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b} &= \sum_{b=0}^{a-r}\sum_{s=0}^{a+b-1}\frac{(-1)^{\boldsymbol{b}}}{a+b}\binom{a+b-s-1}{2b}\binom{s}{r-1}\binom{2b}{b}\\
&= \sum_{s=r-1}^{2a-r-1}\binom{s}{r-1}\sum_{b=0}^{a-s-1}\frac{(-1)^{\boldsymbol{b}}}{a+b}\binom{a+b-s-1}{2b}\binom{2b}{b}\\
&= \sum_{s=r-1}^{a-1}\binom{s}{r-1}\sum_{b=0}^{a-s-1}\frac{(-1)^{\boldsymbol{b}}}{a+b}\binom{a+b-s-1}{2b}\binom{2b}{b}\\
&= \sum_{s=r-1}^{a-1}\binom{s}{r-1}\frac{(-1)^{\boldsymbol{a+s+1}}(a-1)!(a-s-1)!}{(2a-s-1)!}\binom{a-1}{a-s-1}
\end{aligned}$$

Simplification of $\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b}$

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\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b} &= \sum_{b=0}^{a-r}\sum_{s=0}^{a+b-1} \frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{s}{r-1}\binom{2b}{b} \\
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&= \sum_{s=r-1}^{a-1}\binom{s}{r-1}\sum_{b=0}^{a-s-1} \frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{2b}{b} \\
&= \sum_{s=r-1}^{a-1}\binom{s}{r-1}\frac{(-1)^{a+s+1}(a-1)!(a-s-1)!}{(2a-s-1)!}\binom{a-1}{a-s-1} \\
&= \frac{(-1)^{a+1}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=r-1}^{a-1}(-1)^s\binom{2a-r}{s-r+1}
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&= \sum_{s=r-1}^{2a-r-1}\binom{s}{r-1}\sum_{b=0}^{a-s-1}\frac{(-1)^{\boldsymbol{b}}}{a+b}\binom{a+b-s-1}{2b}\binom{2b}{b} \\
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&= \sum_{s=r-1}^{a-1}\binom{s}{r-1}\frac{(-1)^{\boldsymbol{a+s+1}}(a-1)!(a-s-1)!}{(2a-s-1)!}\binom{a-1}{a-s-1} \\
&= \frac{(-1)^{\boldsymbol{a+1}}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=r-1}^{a-1}(-1)^s\binom{2a-r}{s-r+1} \\
(s-r+1 \to s) \qquad &= \frac{(-1)^{\boldsymbol{a+r}}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=0}^{a-r}(-1)^s\binom{2a-r}{s}
\end{aligned}$$

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$$
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&= \frac{(-1)^{\boldsymbol{a+1}}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=r-1}^{a-1}(-1)^s\binom{2a-r}{s-r+1} \\
(s-r+1 \to s) \qquad &= \frac{(-1)^{\boldsymbol{a+r}}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=0}^{a-r}(-1)^s\binom{2a-r}{s} \\
&= \frac{(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\binom{2a-r-1}{a-r}
\end{aligned}
$$

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$$\begin{aligned}
\sum_{b=0}^{a-r}(-1)^b\frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b} &= \sum_{b=0}^{a-r}\sum_{s=0}^{a+b-1}\frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{s}{r-1}\binom{2b}{b}\\
&= \sum_{s=r-1}^{2a-r-1}\binom{s}{r-1}\sum_{b=0}^{a-s-1}\frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{2b}{b}\\
&= \sum_{s=r-1}^{a-1}\binom{s}{r-1}\sum_{b=0}^{a-s-1}\frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{2b}{b}\\
&= \sum_{s=r-1}^{a-1}\binom{s}{r-1}\frac{(-1)^{a+s+1}(a-1)!(a-s-1)!}{(2a-s-1)!}\binom{a-1}{a-s-1}\\
&= \frac{(-1)^{a+1}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=r-1}^{a-1}(-1)^s\binom{2a-r}{s-r+1}\\
(s-r+1\to s)\quad &= \frac{(-1)^{a+r}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=0}^{a-r}(-1)^s\binom{2a-r}{s}\\
&= \frac{(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\binom{2a-r-1}{a-r}\\
&= \frac{1}{2a-r}\binom{a-1}{a-r}
\end{aligned}$$

Plugging in and using the identity

$\sum_{k=0}^{m}(-2)^k \frac{2m+1}{2m-k+1}\binom{m}{k} = \frac{(-1)^m 2^{2m}}{\binom{2m}{m}}$, $(m \in \mathbb{N}_0)$, (Concrete Mathematics)

we get

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{a=2}^{d}(-1)^d a\binom{d}{a}\binom{d+a}{d}\left(\frac{(-2)^{a-2}}{a} - \sum_{r=1}^{a}(-2)^{r-2}\frac{1}{2a-r}\binom{a-1}{a-r}\right)$$

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$$= \sum_{a=2}^{d}(-1)^{d+1} a\binom{d}{a}\binom{d+a}{d}\sum_{r=1}^{a-1}(-2)^{r-2}\frac{1}{2a-r}\binom{a-1}{a-r}$$

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$$= \sum_{a=2}^{d}(-1)^{d+1} a\binom{d}{a}\binom{d+a}{d}\left(\sum_{r=0}^{a-1}(-2)^{r-1}\frac{1}{2a-r-1}\binom{a-1}{r} - (-2)^{a-2}\frac{1}{a}\right)$$

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$$= \sum_{a=2}^{d}(-1)^{d+a+1} a\binom{d}{a}\binom{d+a}{d}\left(2^{2a-3}\frac{1}{2a-1}\frac{1}{\binom{2a-2}{a-1}} - 2^{a-2}\frac{1}{a}\right)$$

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$\sum_{k=0}^{m}(-2)^k \frac{2m+1}{2m-k+1}\binom{m}{k} = \frac{(-1)^m 2^{2m}}{\binom{2m}{m}}$, $(m \in \mathbb{N}_0)$, (Concrete Mathematics)

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$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{a=2}^{d}(-1)^d a\binom{d}{a}\binom{d+a}{d}\left(\frac{(-2)^{a-2}}{a} - \sum_{r=1}^{a}(-2)^{r-2}\frac{1}{2a-r}\binom{a-1}{a-r}\right)$$

$$= \sum_{a=2}^{d}(-1)^{d+1} a\binom{d}{a}\binom{d+a}{d}\sum_{r=1}^{a-1}(-2)^{r-2}\frac{1}{2a-r}\binom{a-1}{a-r} \qquad (r-1 \to r)$$

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$$= \sum_{a=2}^{d}(-1)^{d+a+1} a\binom{d}{a}\binom{d+a}{d}\left(2^{2a-3}\frac{1}{2a-1}\frac{1}{\binom{2a-2}{a-1}} - 2^{a-2}\frac{1}{a}\right)$$

$$= \sum_{a=2}^{d}(-1)^{d+a}\binom{d}{a}\binom{d+a}{d}\left(2^{a-2} - 2^{2a-2}\frac{1}{\binom{2a}{a}}\right)$$

Plugging in and using the identity

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$$\begin{aligned}
\frac{v_d^{(1)}}{v_d^{(0)}} &= \sum_{a=2}^{d}(-1)^d a\binom{d}{a}\binom{d+a}{d}\left(\frac{(-2)^{a-2}}{a} - \sum_{r=1}^{a}(-2)^{r-2}\frac{1}{2a-r}\binom{a-1}{a-r}\right)\\
&= \sum_{a=2}^{d}(-1)^{d+1} a\binom{d}{a}\binom{d+a}{d}\sum_{r=1}^{a-1}(-2)^{r-2}\frac{1}{2a-r}\binom{a-1}{a-r} \qquad (r-1 \to r)\\
&= \sum_{a=2}^{d}(-1)^{d+1} a\binom{d}{a}\binom{d+a}{d}\left(\sum_{r=0}^{a-1}(-2)^{r-1}\frac{1}{2a-r-1}\binom{a-1}{r} - (-2)^{a-2}\frac{1}{a}\right)\\
&= \sum_{a=2}^{d}(-1)^{d+a+1} a\binom{d}{a}\binom{d+a}{d}\left(2^{2a-3}\frac{1}{2a-1}\frac{1}{\binom{2a-2}{a-1}} - 2^{a-2}\frac{1}{a}\right)\\
&= \sum_{a=2}^{d}(-1)^{d+a}\binom{d}{a}\binom{d+a}{d}\left(2^{a-2} - 2^{2a-2}\frac{1}{\binom{2a}{a}}\right)\\
&= \sum_{a=2}^{d}(-1)^{d+a}2^{a-2}\binom{d+a}{2a}\left(\binom{2a}{a} - 2^a\right) \qquad \left(\Rightarrow \frac{v_d^{(1)}}{v_d^{(0)}} \in \mathbb{Z}\right)\\
&= (-1)^d\frac{P_d(-3)-\rho_d(-4)}{4}
\end{aligned}$$

where $P_d(x) = \sum_{k=0}^{d}\binom{d+k}{d-k}\binom{2k}{k}\left(\frac{x-1}{2}\right)^k$ (Legendre polynomial)

$\rho_d(x) = \sum_{k=0}^{d}\binom{d+k}{d-k}x^k$ (associated Legendre polynomial)

Since ρ_d satisfies the recursive formula

$$\begin{aligned}\rho_d(x) =& (x+2)\rho_{d-1}(x) - \rho_{d-2}(x)\\ \rho_0(x) =& 0\\ \rho_1(x) =& x+1\end{aligned}$$

we get $(-1)^d \rho_d(-4) = 2d+1$

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Since P_d satisfies the recursive formula

$$dP_d(x) = (2d-1)xP_{d-1}(x) - (d-1)P_{d-2}(x)$$
$$P_0(x) = 1,\ P_1(x) = x$$

we get that

$$(d+1)\frac{v_d^{(1)}}{v_d^{(0)}} = 3(2d-1)\frac{v_{d-1}^{(1)}}{v_{d-1}^{(0)}} - d\frac{v_{d-2}^{(1)}}{v_{d-2}^{(0)}} + 2d(d+1)$$
$$\frac{v_0^{(1)}}{v_0^{(0)}} = 0,\ \frac{v_1^{(1)}}{v_1^{(0)}} = 0$$