A fast algorithm for simultaneous computation of matching-polynomials of simple graphs

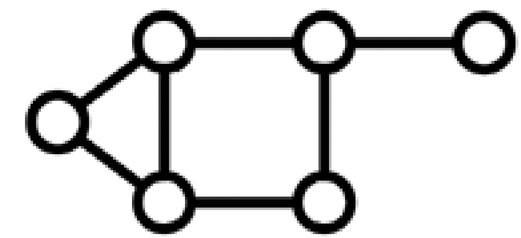
Mario Weitzer

July 8, 2010

Definitions

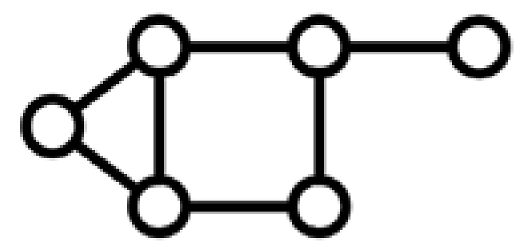
Definitions

Simple Graph:
Undirected graph without loops or multiple edges

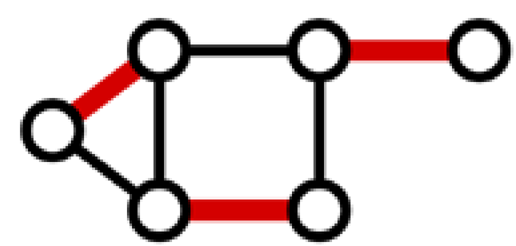


Definitions

Simple Graph:
Undirected graph without loops or multiple edges



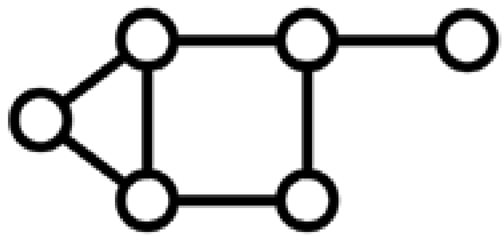
Matching of a graph:
Subset of the edge set such that no two edges share a common vertex (Independet edge set)



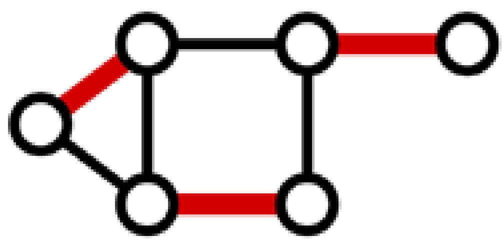
k - Matching of a graph:
Matching of cardinality k

Definitions

Simple Graph:
Undirected graph without loops or multiple edges



Matching of a graph:
Subset of the edge set such that no two edges share a common vertex (Independet edge set)

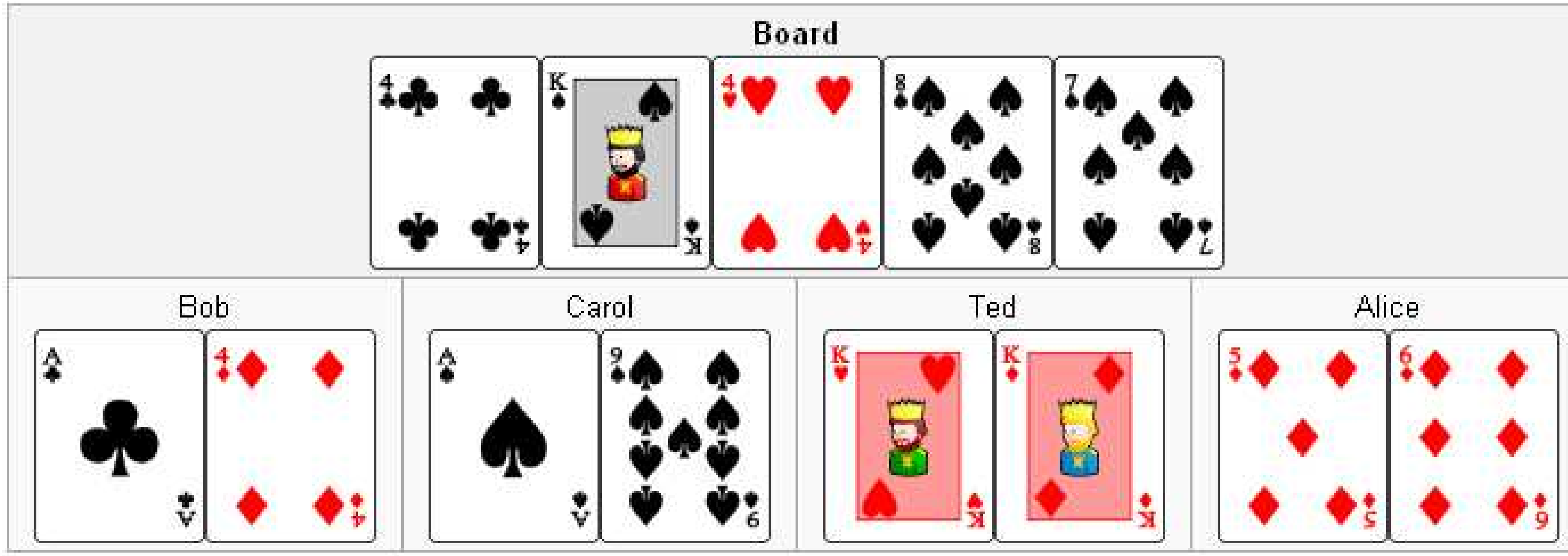


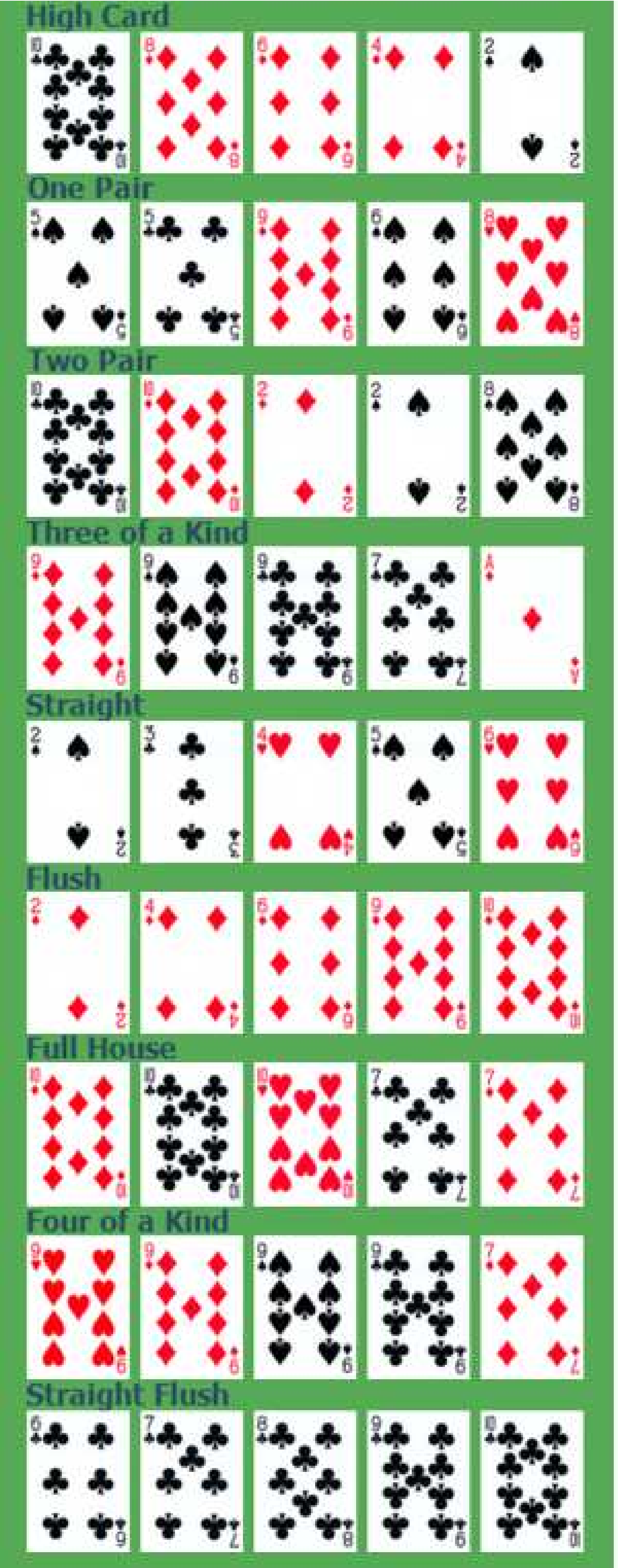
k - Matching of a graph:
Matching of cardinality k

Matching polynomial of a graph:
Coefficient of x^k is number of k - Matchings

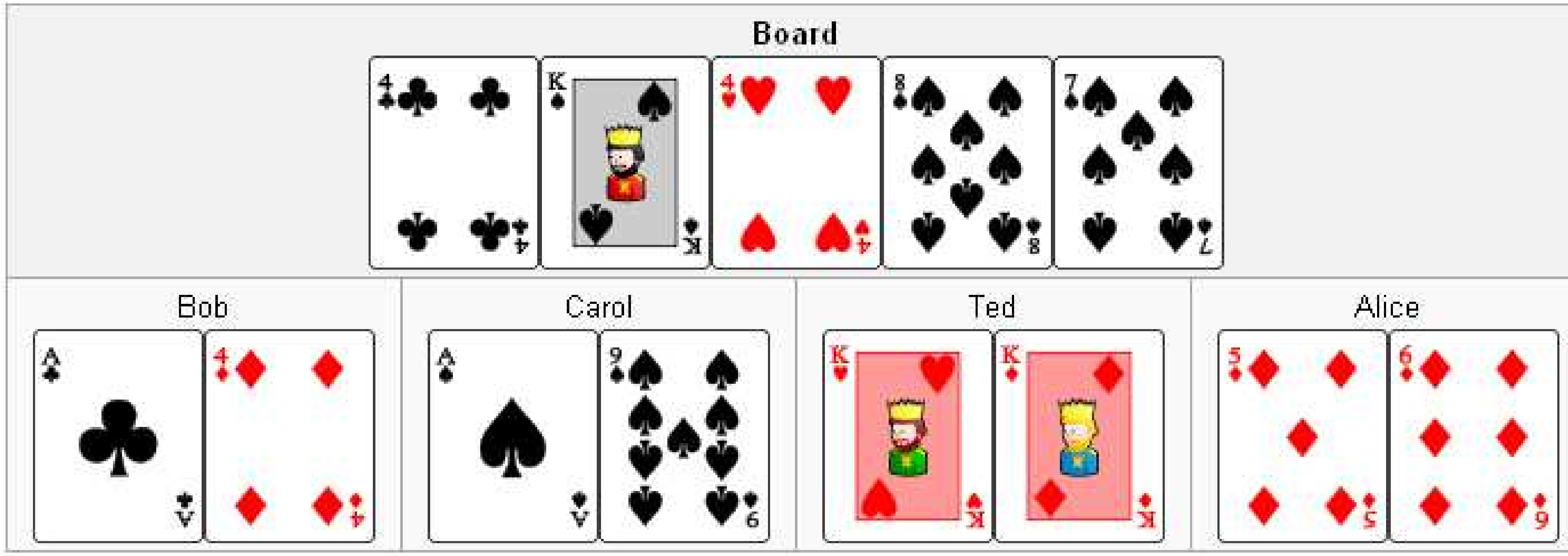
Connection to probabilities in Texas Hold'em Poker?

Sample showdown

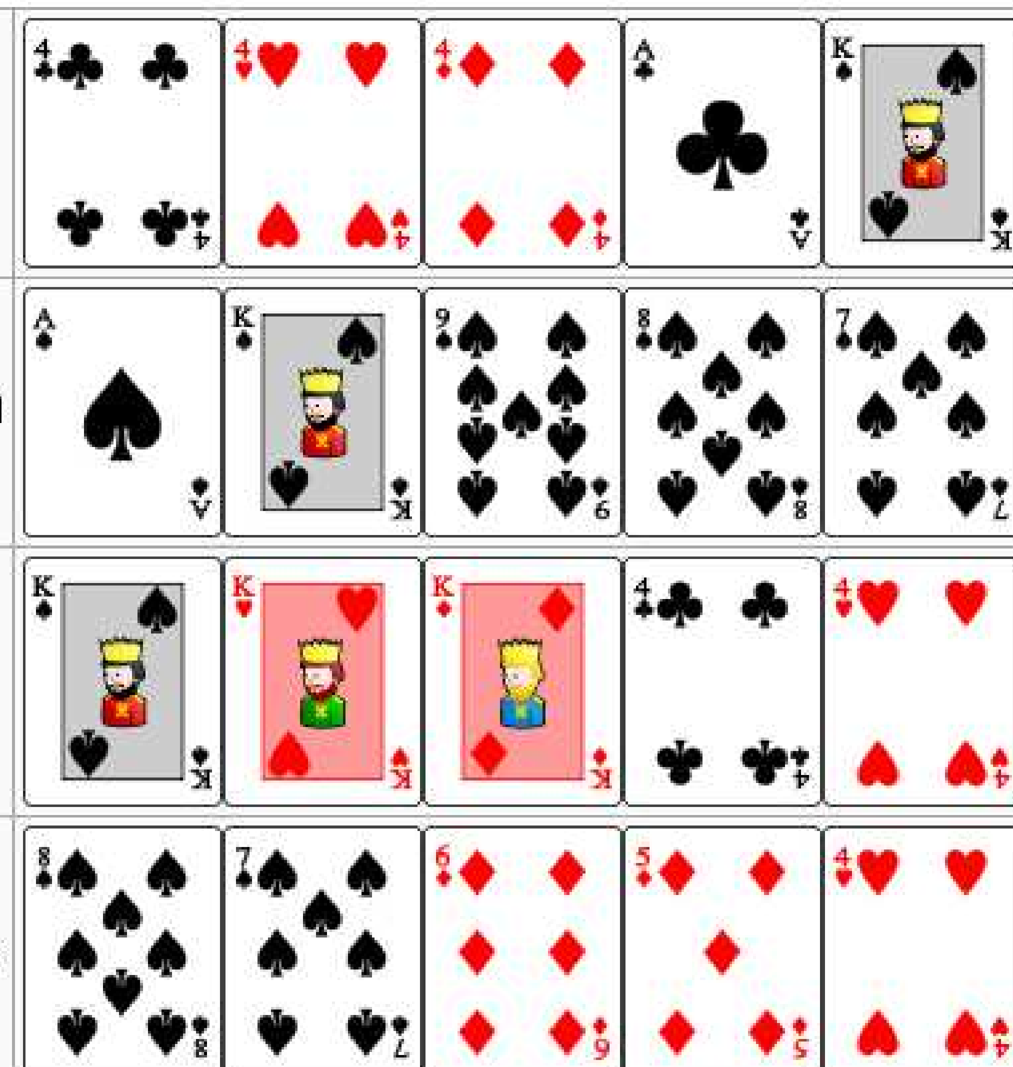




Sample showdown

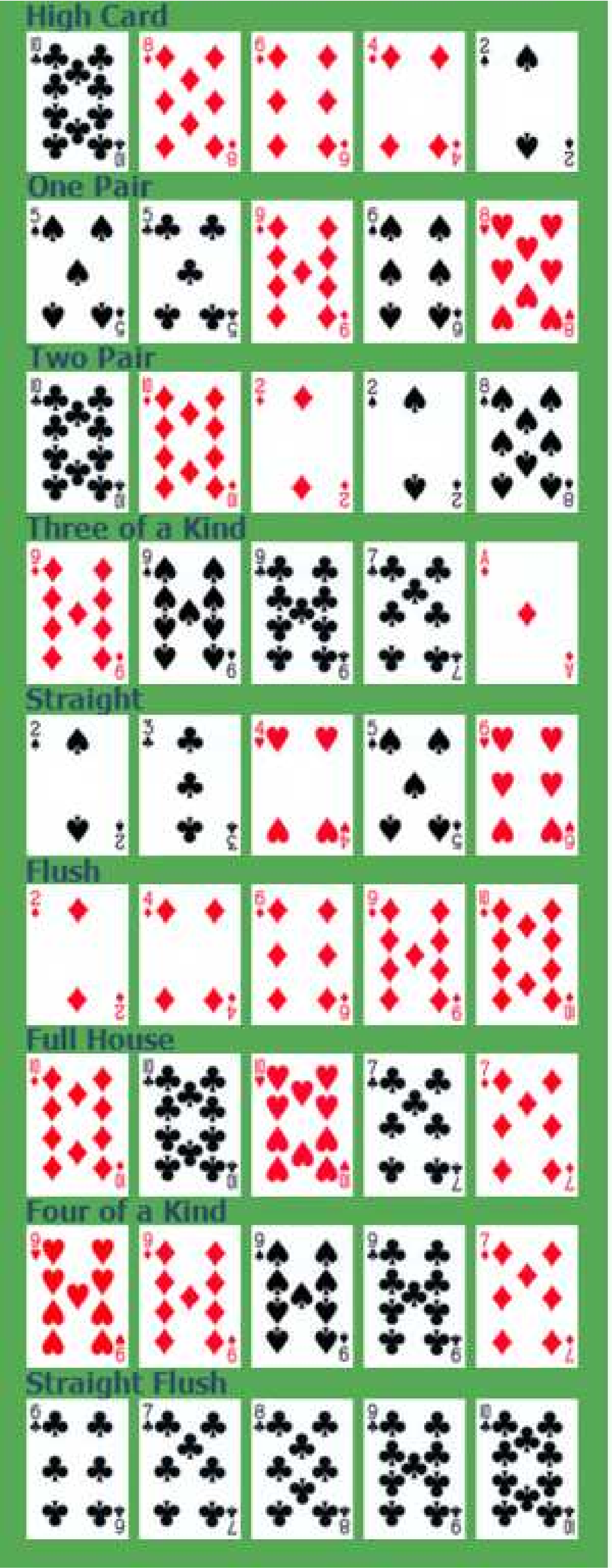


Each player plays the best 5-card hand they can make with the seven cards available. They have

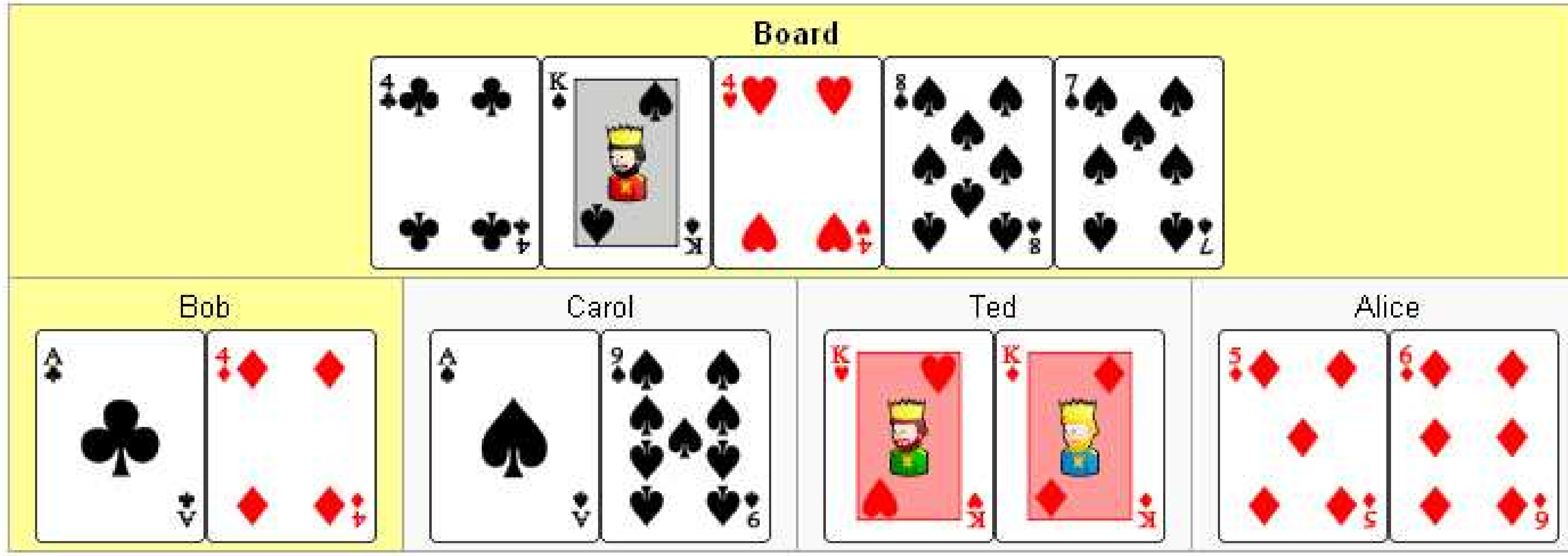


Bob	Three fours, with ace, king kickers
Carol	Ace-high flush
Ted	Full house, kings full of fours
Alice	8-high straight

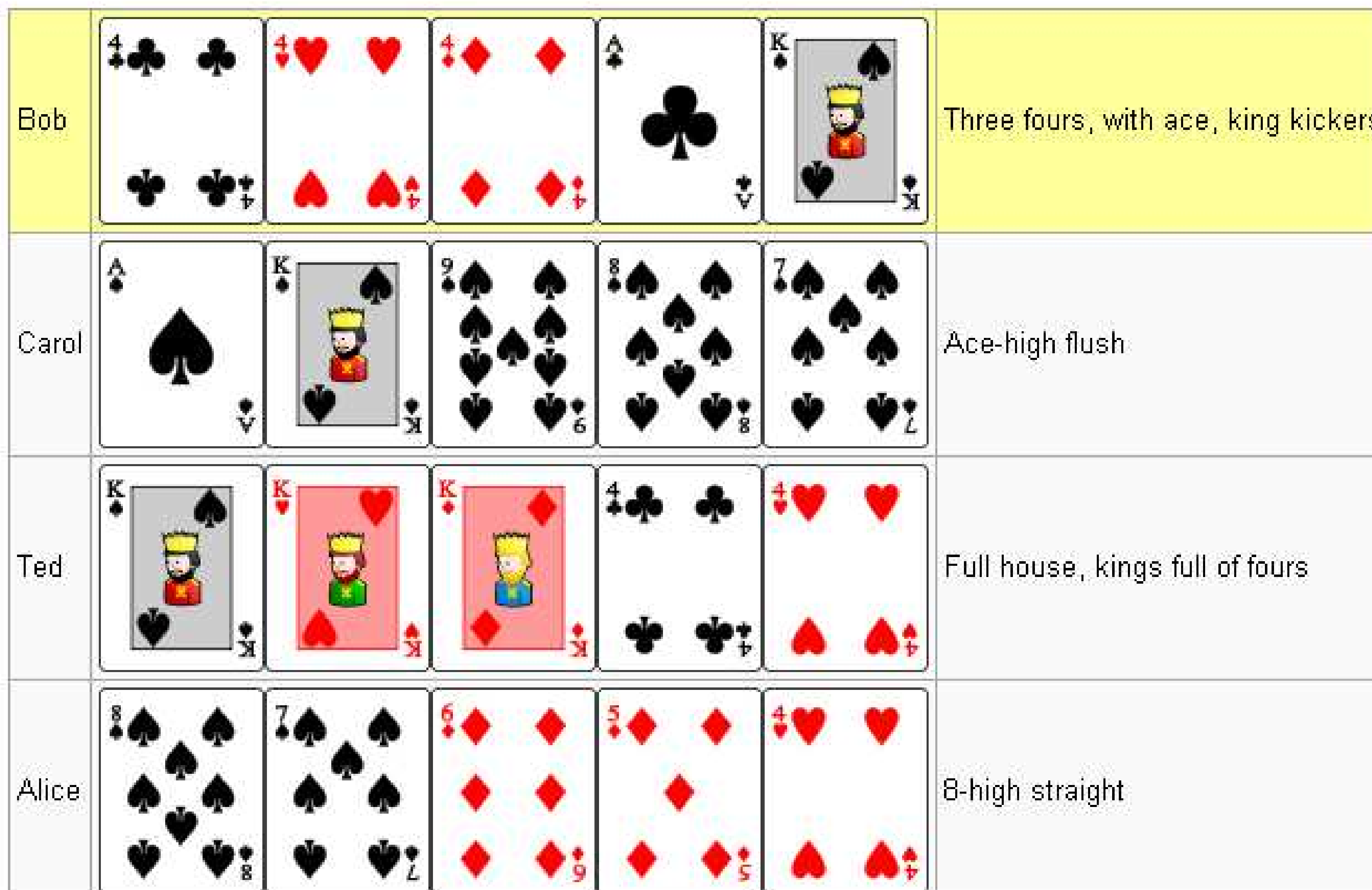
In this case, Ted's full house is the best hand, with Carol in 2nd, Alice in 3rd and Bob last.



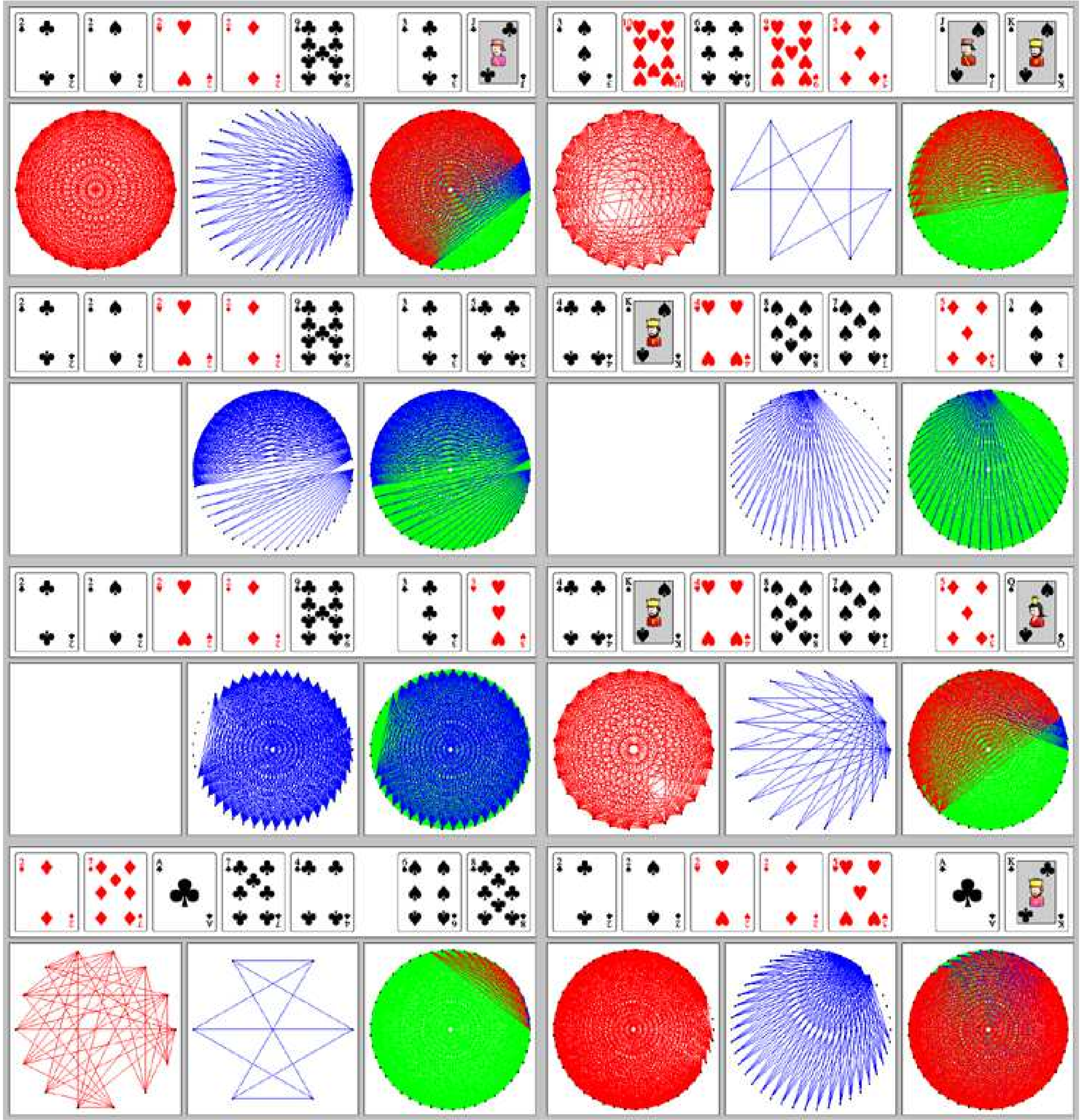
Sample showdown



Each player plays the best 5-card hand they can make with the seven cards available. They have



In this case, Ted's full house is the best hand, with Carol in 2nd, Alice in 3rd and Bob last.



The algorithm

The algorithm

Applicable to general simple graphs

The algorithm

Applicable to general simple graphs

Calculates coefficients one at a time

The algorithm

Applicable to general simple graphs

Calculates coefficients one at a time

Brute force algorithm

The algorithm

Applicable to general simple graphs

Calculates coefficients one at a time

Brute force algorithm

Extensive use of three main resources of computers

- Computational Power
- Main Memory (RAM)
- Hard Disc Space

The algorithm

Applicable to general simple graphs

Calculates coefficients one at a time

Brute force algorithm

Extensive use of three main resources of computers

- Computational Power
- Main Memory (RAM)
- Hard Disc Space

Achieve

- Maximal factorization within a single and among several graphs
- Reduction of computation time

Performance

Performance

"What experience I have suggests that it could be difficult to develop something useful for graphs on 12 vertices. So getting the first nine coefficients of just one graph on 45 vertices seems impossible at the moment." - Chris Godsil

Performance

"What experience I have suggests that it could be difficult to develop something useful for graphs on 12 vertices. So getting the first nine coefficients of just one graph on 45 vertices seems impossible at the moment." - Chris Godsil

An algorithm for calculating the independence and vertex-cover polynomials of a graph
Applied Mathematics and Computation 190, 2007
Gordon G. Cash

Performance

"What experience I have suggests that it could be difficult to develop something useful for graphs on 12 vertices. So getting the first nine coefficients of just one graph on 45 vertices seems impossible at the moment." - Chris Godsil

An algorithm for calculating the independence and vertex-cover polynomials of a graph
Applied Mathematics and Computation 190, 2007
Gordon G. Cash

Computed first five coefficients of 151215 graphs on 45 vertices in less than one week using

- 4 x 2.67 GHz Quad-Core CPU
- 2 GB RAM
- 70 GB hard disc space

Performance

"What experience I have suggests that it could be difficult to develop something useful for graphs on 12 vertices. So getting the first nine coefficients of just one graph on 45 vertices seems impossible at the moment." - Chris Godsil

An algorithm for calculating the independence and vertex-cover polynomials of a graph
Applied Mathematics and Computation 190, 2007
Gordon G. Cash

Computed first five coefficients of 151215 graphs on 45 vertices in less than one week using

- 4 x 2.67 GHz Quad-Core CPU
- 2 GB RAM
- 70 GB hard disc space

Would compute first nine coefficients in less than 60000 CPU-hours using

- 4 GB RAM per core
- 15 TB hard disc space

Performance

"What experience I have suggests that it could be difficult to develop something useful for graphs on 12 vertices. So getting the first nine coefficients of just one graph on 45 vertices seems impossible at the moment." - Chris Godsil

An algorithm for calculating the independence and vertex-cover polynomials of a graph
Applied Mathematics and Computation 190, 2007
Gordon G. Cash

Computed first five coefficients of 151215 graphs on 45 vertices in less than one week using

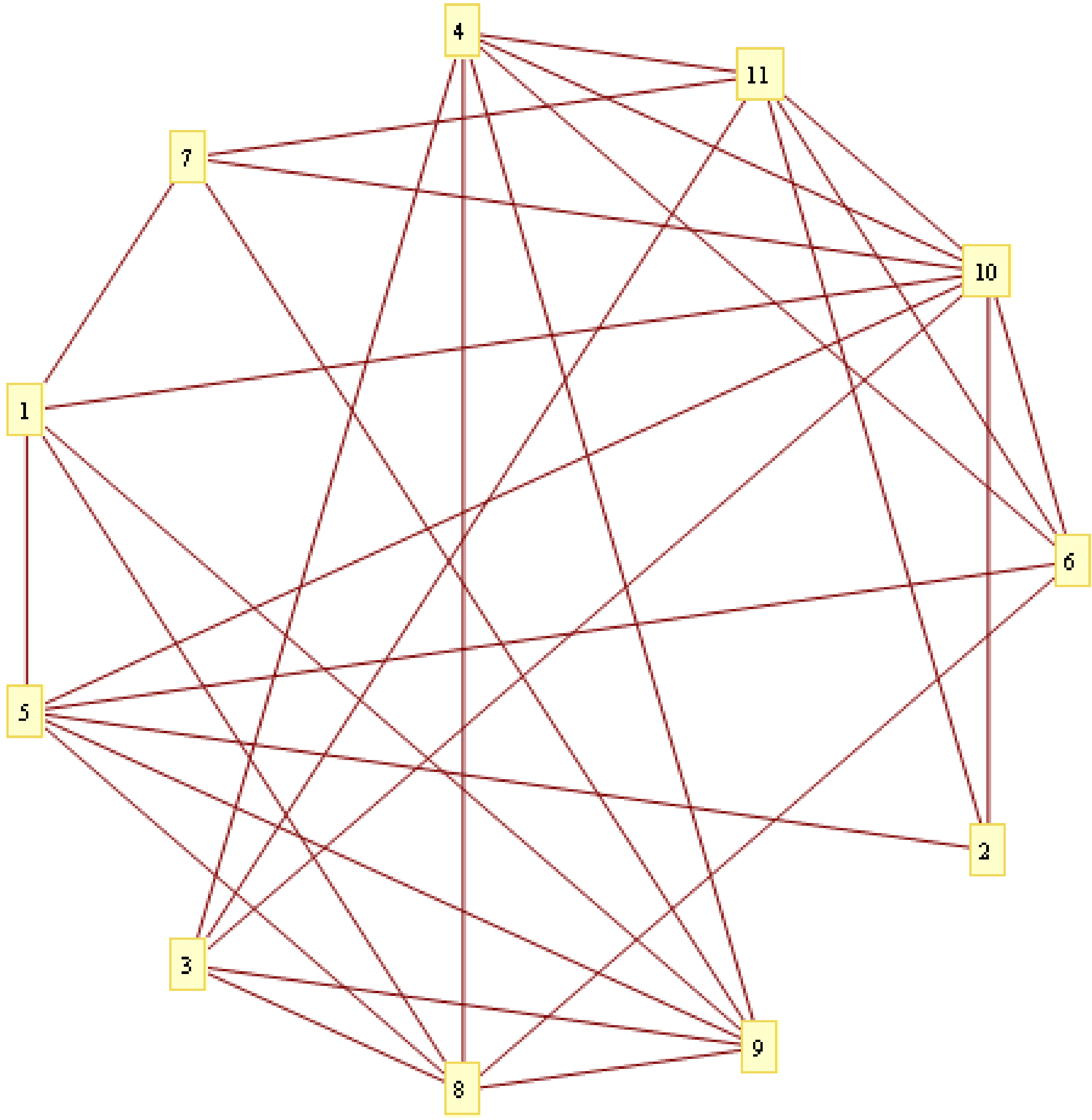
- 4 x 2.67 GHz Quad-Core CPU
- 2 GB RAM
- 70 GB hard disc space

Would compute first nine coefficients in less than 60000 CPU-hours using

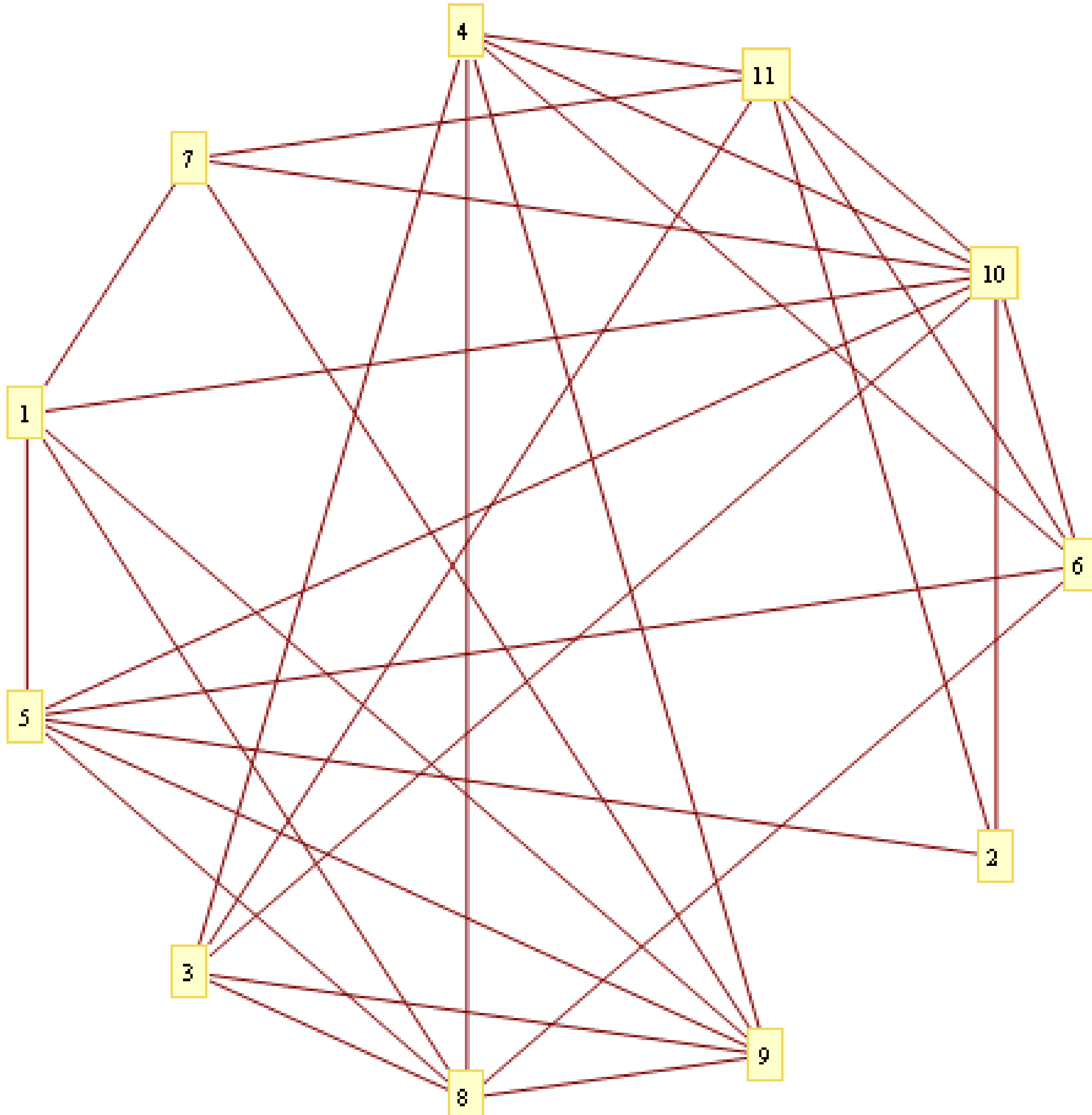
- 4 GB RAM per core
- 15 TB hard disc space

Open question: Complexity

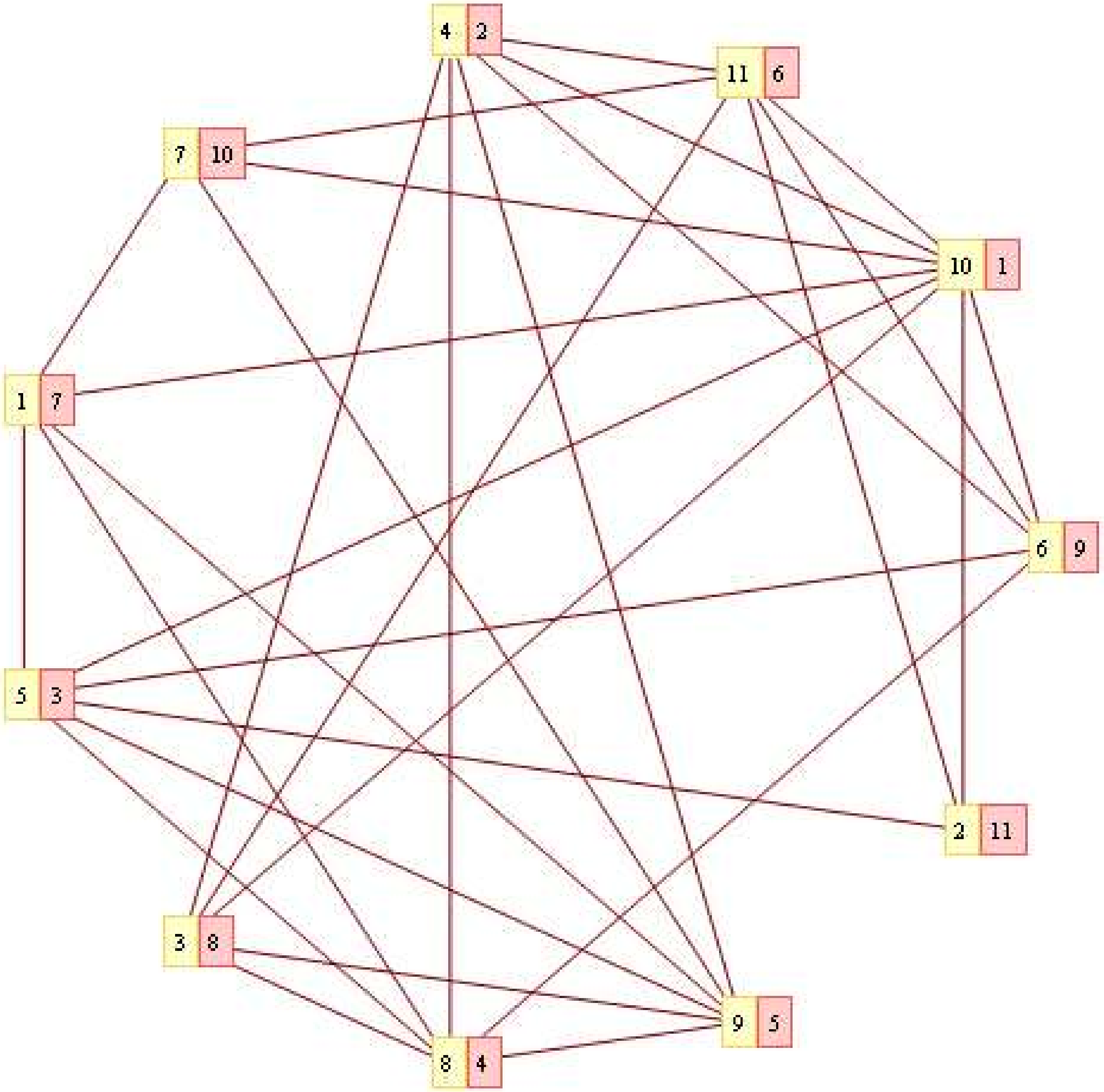
Step 1: Preparation of graphs



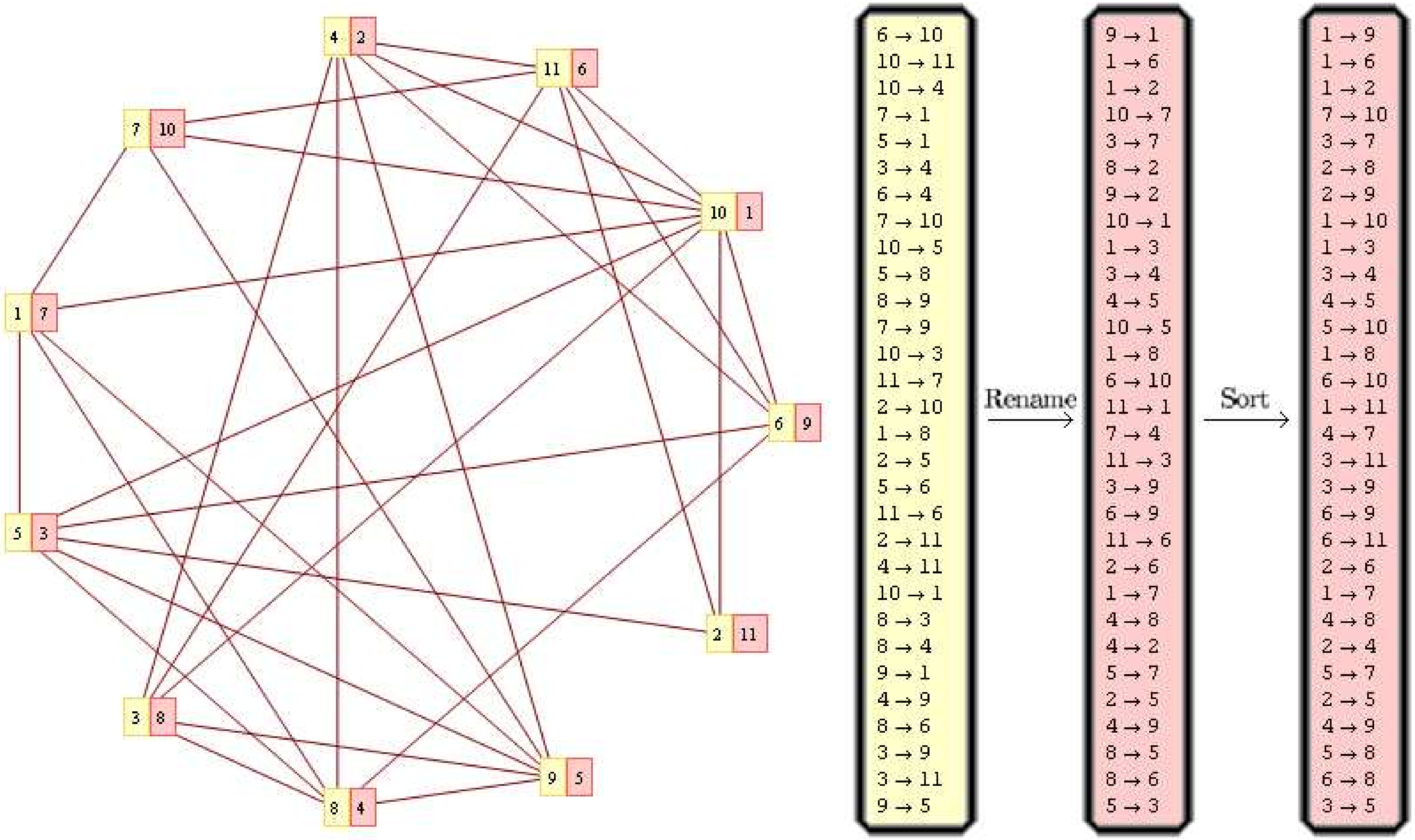
4
11
7
10
1
6
5
2
3
9
8



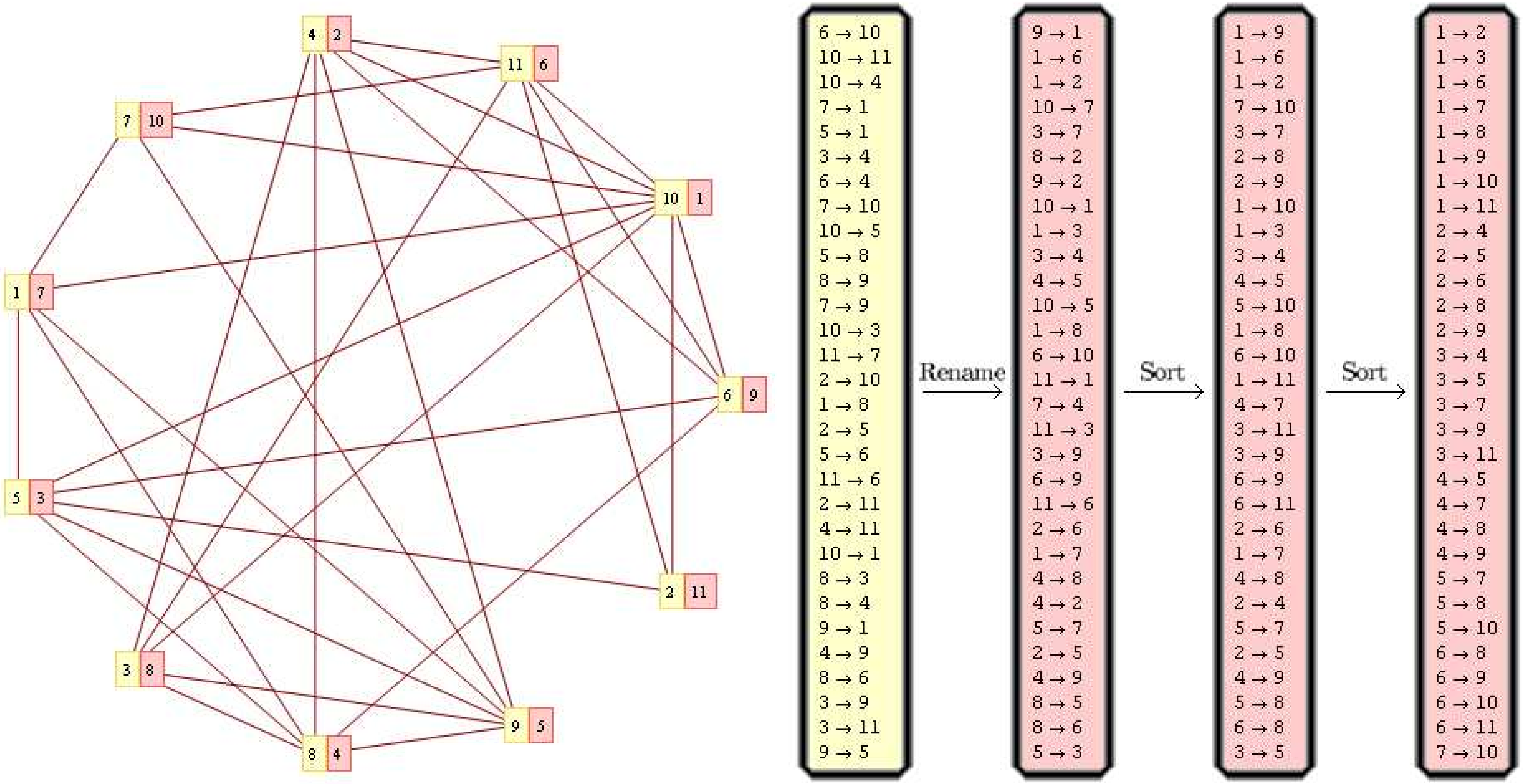
6 → 10
10 → 11
10 → 4
7 → 1
5 → 1
3 → 4
6 → 4
7 → 10
10 → 5
5 → 8
8 → 9
7 → 9
10 → 3
11 → 7
2 → 10
1 → 8
2 → 5
5 → 6
11 → 6
2 → 11
4 → 11
10 → 1
8 → 3
8 → 4
9 → 1
4 → 9
8 → 6
3 → 9
3 → 11
9 → 5

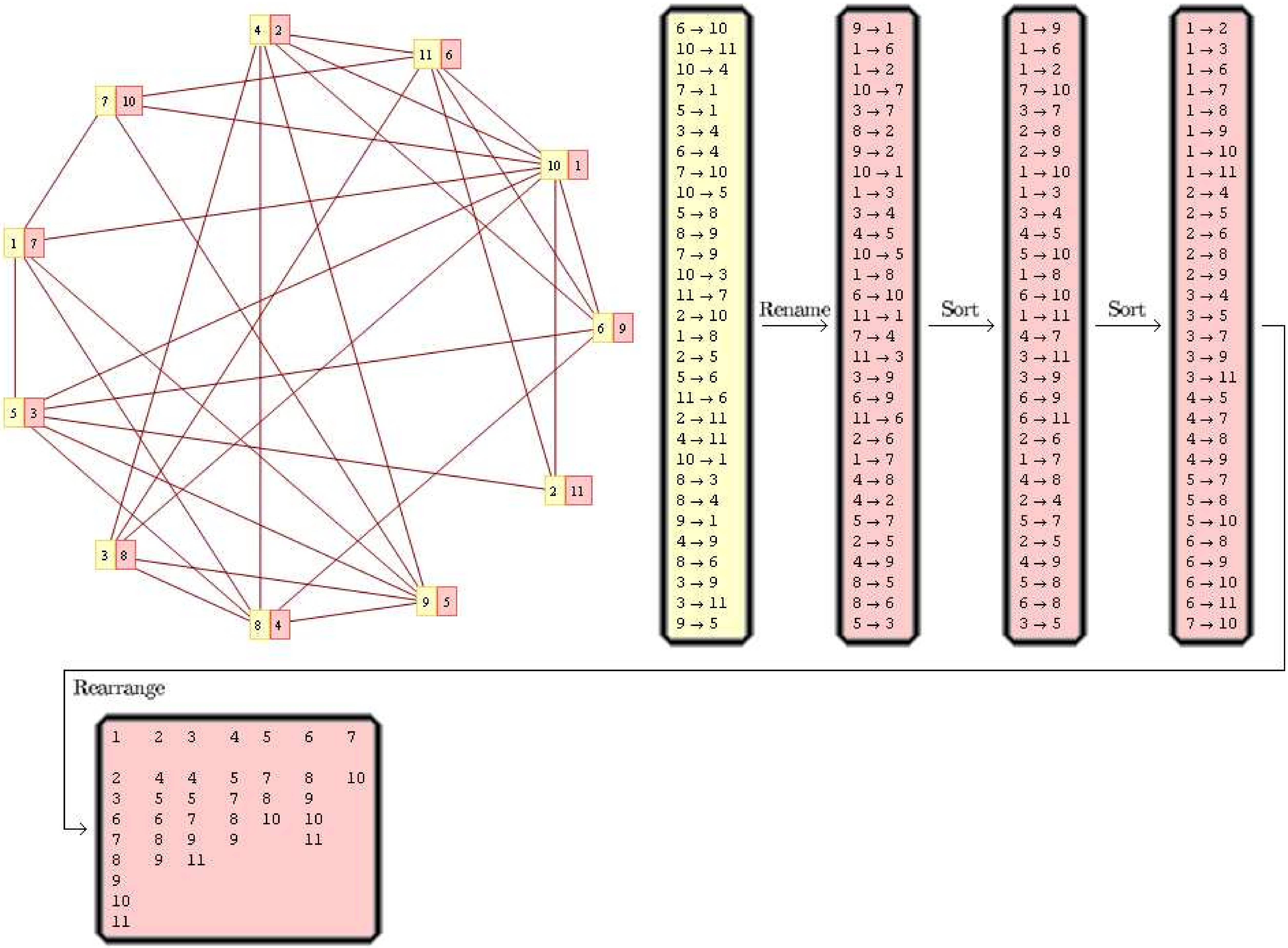


	Rename	
6 → 10		9 → 1
10 → 11		1 → 6
10 → 4		1 → 2
7 → 1		10 → 7
5 → 1		3 → 7
3 → 4		8 → 2
6 → 4		9 → 2
7 → 10		10 → 1
10 → 5		1 → 3
5 → 8		3 → 4
8 → 9		4 → 5
7 → 9		10 → 5
10 → 3		1 → 8
11 → 7		6 → 10
2 → 10		11 → 1
1 → 8		7 → 4
2 → 5		11 → 3
5 → 6		3 → 9
11 → 6		6 → 9
2 → 11		11 → 6
4 → 11		2 → 6
10 → 1		1 → 7
8 → 3		4 → 8
8 → 4		4 → 2
9 → 1		5 → 7
4 → 9		2 → 5
8 → 6		4 → 9
3 → 9		8 → 5
3 → 11		8 → 6
9 → 5		5 → 3

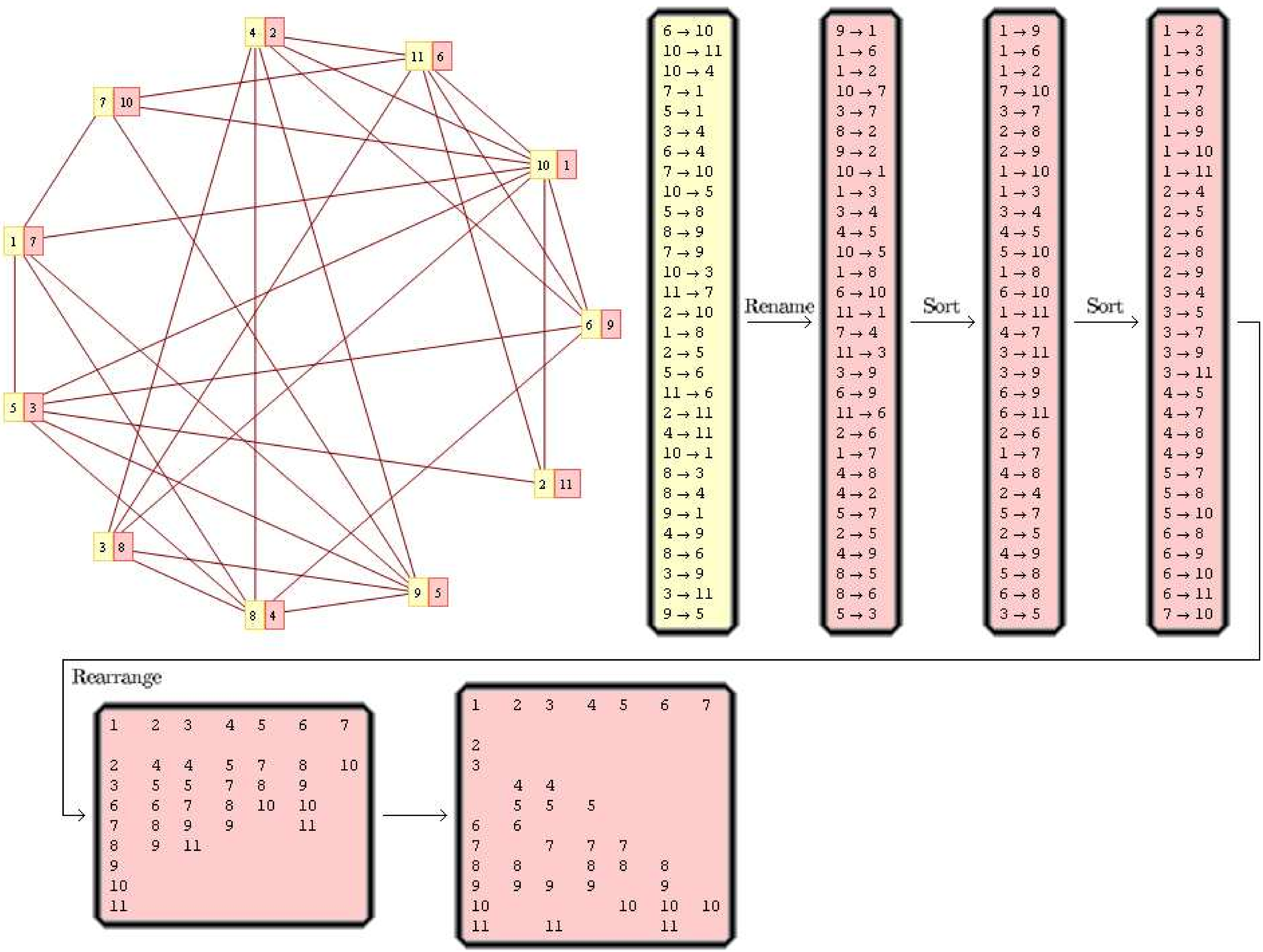


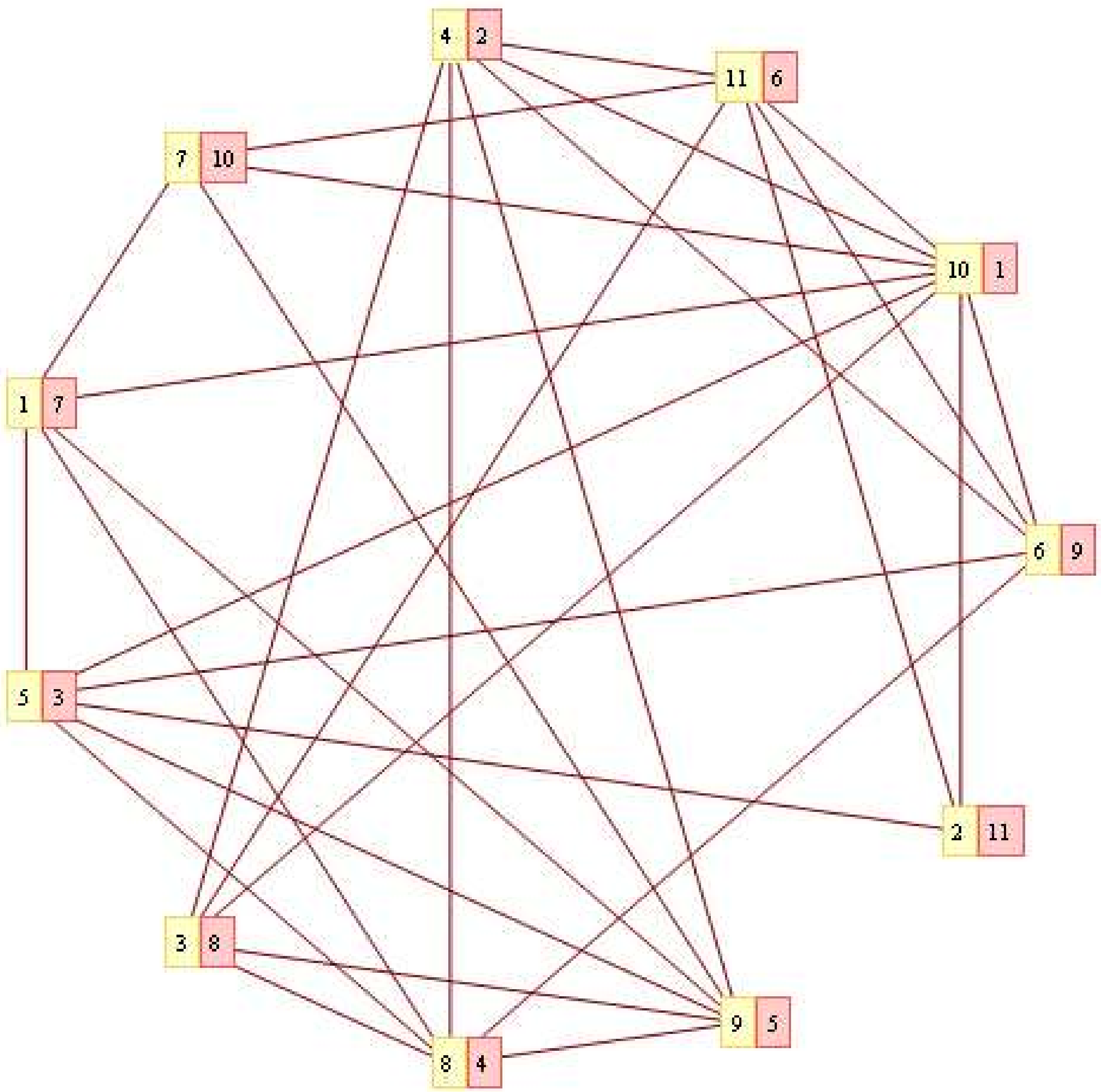
4 2
11 6
7 10
10 1
1 7
6 9
5 3
2 11
3 8
9 5
8 4
6 → 10
10 → 11
10 → 4
7 → 1
5 → 1
3 → 4
6 → 4
7 → 10
10 → 5
5 → 8
8 → 9
7 → 9
10 → 3
11 → 7
2 → 10
1 → 8
2 → 5
5 → 6
11 → 6
2 → 11
4 → 11
10 → 1
8 → 3
8 → 4
9 → 1
4 → 9
8 → 6
3 → 9
3 → 11
9 → 5
Rename
9 → 1
1 → 6
1 → 2
10 → 7
3 → 7
8 → 2
9 → 2
10 → 1
1 → 3
3 → 4
4 → 5
10 → 5
1 → 8
6 → 10
11 → 1
7 → 4
11 → 3
3 → 9
6 → 9
11 → 6
2 → 6
1 → 7
4 → 8
4 → 2
5 → 7
2 → 5
4 → 9
8 → 5
8 → 6
5 → 3
Sort
1 → 9
1 → 6
1 → 2
7 → 10
3 → 7
2 → 8
2 → 9
1 → 10
1 → 3
3 → 4
4 → 5
5 → 10
1 → 8
6 → 10
1 → 11
4 → 7
3 → 11
3 → 9
6 → 9
6 → 11
2 → 6
1 → 7
4 → 8
2 → 4
5 → 7
2 → 5
4 → 9
5 → 8
6 → 8
3 → 5


6 → 10
10 → 11
10 → 4
7 → 1
5 → 1
3 → 4
6 → 4
7 → 10
10 → 5
5 → 8
8 → 9
7 → 9
10 → 3
11 → 7
2 → 10
1 → 8
2 → 5
5 → 6
11 → 6
2 → 11
4 → 11
10 → 1
8 → 3
8 → 4
9 → 1
4 → 9
8 → 6
3 → 9
3 → 11
9 → 5
Rename
9 → 1
1 → 6
1 → 2
10 → 7
3 → 7
8 → 2
9 → 2
10 → 1
1 → 3
3 → 4
4 → 5
10 → 5
1 → 8
6 → 10
11 → 1
7 → 4
11 → 3
3 → 9
6 → 9
11 → 6
2 → 6
1 → 7
4 → 8
4 → 2
5 → 7
2 → 5
4 → 9
8 → 5
8 → 6
5 → 3
Sort
1 → 9
1 → 6
1 → 2
7 → 10
3 → 7
2 → 8
2 → 9
1 → 10
1 → 3
3 → 4
4 → 5
5 → 10
1 → 8
6 → 10
1 → 11
4 → 7
3 → 11
3 → 9
6 → 9
6 → 11
2 → 6
1 → 7
4 → 8
2 → 4
5 → 7
2 → 5
4 → 9
5 → 8
6 → 8
3 → 5
Sort
1 → 2
1 → 3
1 → 6
1 → 7
1 → 8
1 → 9
1 → 10
1 → 11
2 → 4
2 → 5
2 → 6
2 → 8
2 → 9
3 → 4
3 → 5
3 → 7
3 → 9
3 → 11
4 → 5
4 → 7
4 → 8
4 → 9
5 → 7
5 → 8
5 → 10
6 → 8
6 → 9
6 → 10
6 → 11
7 → 10



4 2
11 6
7 10
10 1
1 7
6 9
5 3
2 11
3 8
8 4
9 5
6 → 10
10 → 11
10 → 4
7 → 1
5 → 1
3 → 4
6 → 4
7 → 10
10 → 5
5 → 8
8 → 9
7 → 9
10 → 3
11 → 7
2 → 10
1 → 8
2 → 5
5 → 6
11 → 6
2 → 11
4 → 11
10 → 1
8 → 3
8 → 4
9 → 1
4 → 9
8 → 6
3 → 9
3 → 11
9 → 5
Rename
9 → 1
1 → 6
1 → 2
10 → 7
3 → 7
8 → 2
9 → 2
10 → 1
1 → 3
3 → 4
4 → 5
10 → 5
1 → 8
6 → 10
11 → 1
7 → 4
11 → 3
3 → 9
6 → 9
11 → 6
2 → 6
1 → 7
4 → 8
4 → 2
5 → 7
2 → 5
4 → 9
8 → 5
8 → 6
5 → 3
Sort
1 → 9
1 → 6
1 → 2
7 → 10
3 → 7
2 → 8
2 → 9
1 → 10
1 → 3
3 → 4
4 → 5
5 → 10
1 → 8
6 → 10
1 → 11
4 → 7
3 → 11
3 → 9
6 → 9
6 → 11
2 → 6
1 → 7
4 → 8
2 → 4
5 → 7
2 → 5
4 → 9
5 → 8
6 → 8
3 → 5
Sort
1 → 2
1 → 3
1 → 6
1 → 7
1 → 8
1 → 9
1 → 10
1 → 11
2 → 4
2 → 5
2 → 6
2 → 8
2 → 9
3 → 4
3 → 5
3 → 7
3 → 9
3 → 11
4 → 5
4 → 7
4 → 8
4 → 9
5 → 7
5 → 8
5 → 10
6 → 8
6 → 9
6 → 10
6 → 11
7 → 10
Rearrange
1 2 3 4 5 6 7
2 4 4 5 7 8 10
3 5 5 7 8 9
6 6 7 8 10 10
7 8 9 9 11
8 9 11
9
10
11


4 2
11 6
7 10
10 1
1 7
6 9
5 3
2 11
3 8
9 5
8 4
6 → 10
10 → 11
10 → 4
7 → 1
5 → 1
3 → 4
6 → 4
7 → 10
10 → 5
5 → 8
8 → 9
7 → 9
10 → 3
11 → 7
2 → 10
1 → 8
2 → 5
5 → 6
11 → 6
2 → 11
4 → 11
10 → 1
8 → 3
8 → 4
9 → 1
4 → 9
8 → 6
3 → 9
3 → 11
9 → 5
Rename
9 → 1
1 → 6
1 → 2
10 → 7
3 → 7
8 → 2
9 → 2
10 → 1
1 → 3
3 → 4
4 → 5
10 → 5
1 → 8
6 → 10
11 → 1
7 → 4
11 → 3
3 → 9
6 → 9
11 → 6
2 → 6
1 → 7
4 → 8
4 → 2
5 → 7
2 → 5
4 → 9
8 → 5
8 → 6
5 → 3
Sort
1 → 9
1 → 6
1 → 2
7 → 10
3 → 7
2 → 8
2 → 9
1 → 10
1 → 3
3 → 4
4 → 5
5 → 10
1 → 8
6 → 10
1 → 11
4 → 7
3 → 11
3 → 9
6 → 9
6 → 11
2 → 6
1 → 7
4 → 8
2 → 4
5 → 7
2 → 5
4 → 9
5 → 8
6 → 8
3 → 5
Sort
1 → 2
1 → 3
1 → 6
1 → 7
1 → 8
1 → 9
1 → 10
1 → 11
2 → 4
2 → 5
2 → 6
2 → 8
2 → 9
3 → 4
3 → 5
3 → 7
3 → 9
3 → 11
4 → 5
4 → 7
4 → 8
4 → 9
5 → 7
5 → 8
5 → 10
6 → 8
6 → 9
6 → 10
6 → 11
7 → 10
Rearrange
1 2 3 4 5 6 7
2 4 4 5 7 8 10
3 5 5 7 8 9
6 6 7 8 10 10
7 8 9 9 11
8 9 11
9
10
11
1 2 3 4 5 6 7
2
3
4 4
5 5 5
6 6
7 7 7 7
8 8 8 8 8
9 9 9 9 9
10 10 10 10
11 11 11



	Rename →	Sort →	Sort →
6 → 10	9 → 1	1 → 9	1 → 2
10 → 11	1 → 6	1 → 6	1 → 3
10 → 4	1 → 2	1 → 2	1 → 6
7 → 1	10 → 7	7 → 10	1 → 7
5 → 1	3 → 7	3 → 7	1 → 8
3 → 4	8 → 2	2 → 8	1 → 9
6 → 4	9 → 2	2 → 9	1 → 10
7 → 10	10 → 1	1 → 10	1 → 11
10 → 5	1 → 3	1 → 3	2 → 4
5 → 8	3 → 4	3 → 4	2 → 5
8 → 9	4 → 5	4 → 5	2 → 6
7 → 9	10 → 5	5 → 10	2 → 8
10 → 3	1 → 8	1 → 8	2 → 9
11 → 7	6 → 10	6 → 10	3 → 4
2 → 10	11 → 1	1 → 11	3 → 5
1 → 8	7 → 4	4 → 7	3 → 7
2 → 5	11 → 3	3 → 11	3 → 9
5 → 6	3 → 9	3 → 9	3 → 11
11 → 6	6 → 9	6 → 9	4 → 5
2 → 11	11 → 6	6 → 11	4 → 7
4 → 11	2 → 6	2 → 6	4 → 8
10 → 1	1 → 7	1 → 7	4 → 9
8 → 3	4 → 8	4 → 8	5 → 7
8 → 4	4 → 2	2 → 4	5 → 8
9 → 1	5 → 7	5 → 7	5 → 10
4 → 9	2 → 5	2 → 5	6 → 8
8 → 6	4 → 9	4 → 9	6 → 9
3 → 9	8 → 5	5 → 8	6 → 10
3 → 11	8 → 6	6 → 8	6 → 11
9 → 5	5 → 3	3 → 5	7 → 10

Rearrange →

1	2	3	4	5	6	7
2	4	4	5	7	8	10
3	5	5	7	8	9	
6	6	7	8	10	10	
7	8	9	9		11	
8	9	11				
9						
10						
11						

→

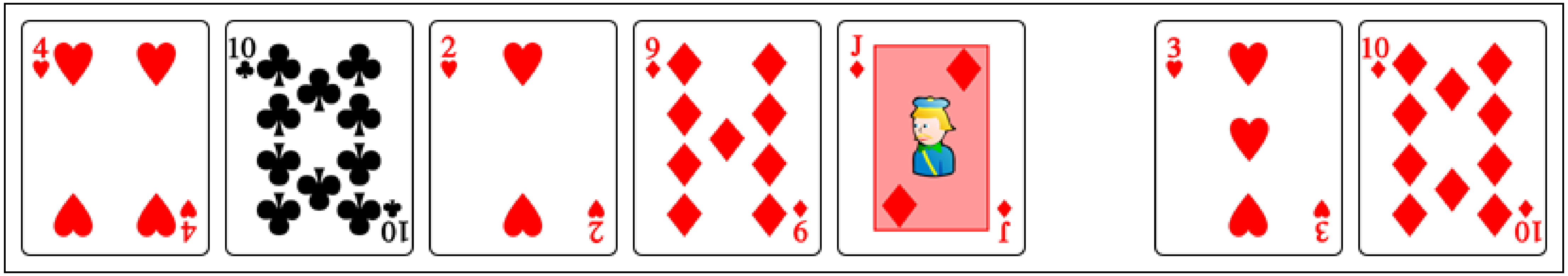
1	2	3	4	5	6	7
2						
3						
	4	4				
	5	5	5			
6	6					
7		7	7	7		
8	8		8	8	8	
9	9	9	9		9	
10				10	10	10
11		11			11	

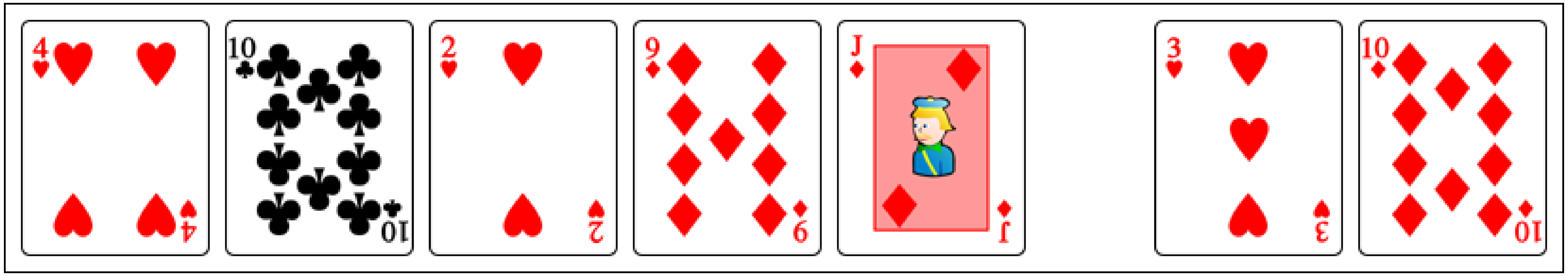
→

	1	2	3	4	5	6	7
2	1	0	0	0	0	0	0
3	1	0	0	0	0	0	0
4	0	1	1	0	0	0	0
5	0	1	1	1	0	0	0
6	1	1	0	0	0	0	0
7	1	0	1	1	1	0	0
8	1	1	0	1	1	1	0
9	1	1	1	1	0	1	0
10	1	0	0	0	1	1	1
11	1	0	1	0	0	1	0

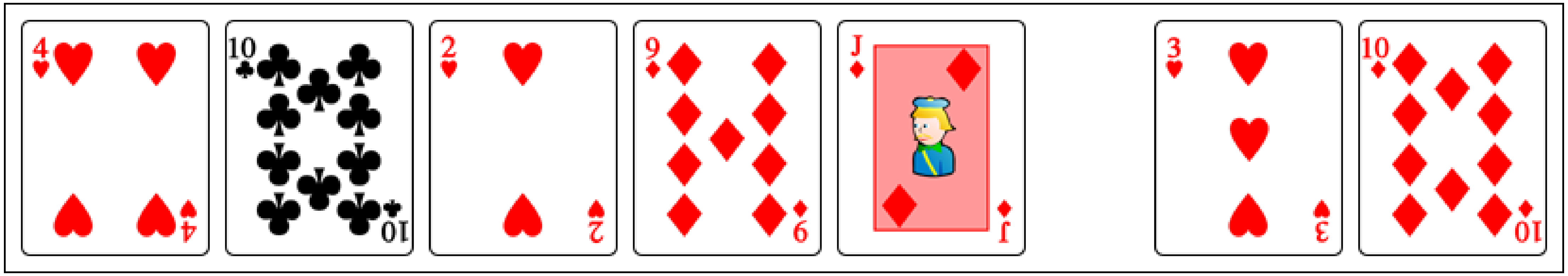
Identify and cancel isomorphic graphs

How many graphs occur in Texas Hold'em Poker?


4
10
2
9
J
3
10

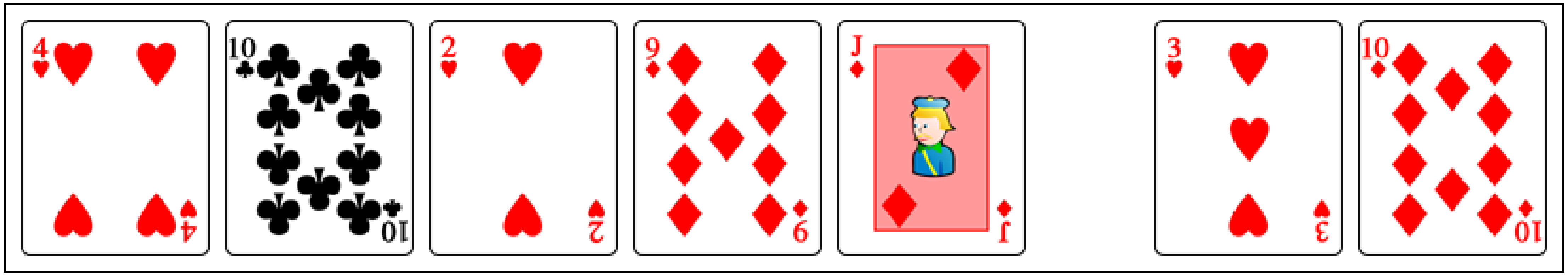


$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = \underline{674274182400}$$



$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = \underline{674274182400}$$

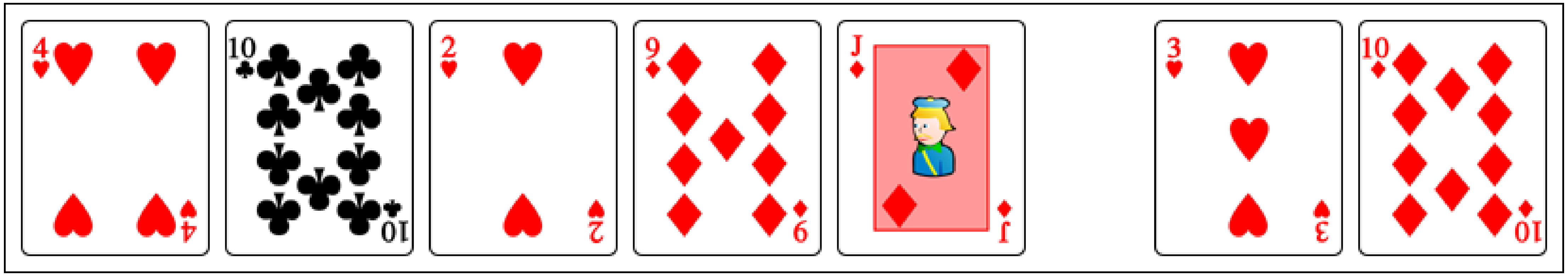
$$\binom{52}{5} \cdot \binom{47}{2} = \underline{2809475760}$$



$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = \underline{674274182400}$$

$$\binom{52}{5} \cdot \binom{47}{2} = \underline{2809475760}$$

$$558883 \cdot 715 = \underline{399601345}$$

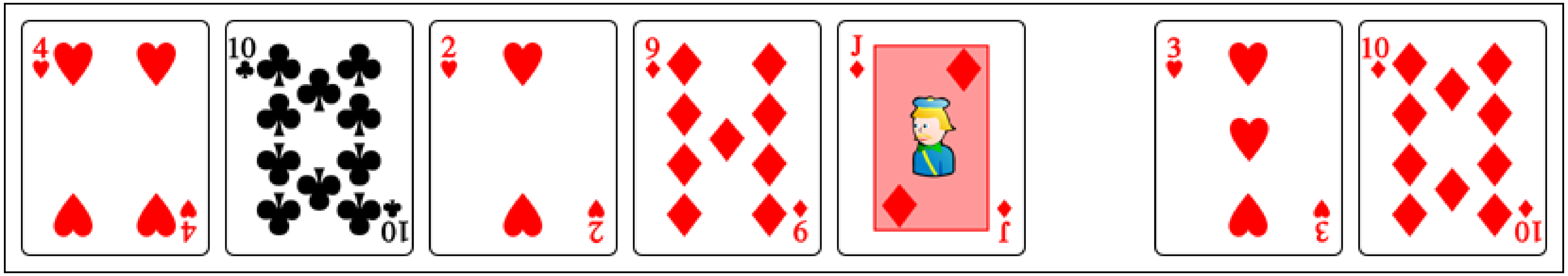


$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = \underline{674274182400}$$

$$\binom{52}{5} \cdot \binom{47}{2} = \underline{2809475760}$$

$$558883 \cdot 715 = \underline{399601345}$$

$$\underline{157506622}$$



$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = \underline{674274182400}$$

$$\binom{52}{5} \cdot \binom{47}{2} = \underline{2809475760}$$

$$558883 \cdot 715 = \underline{399601345}$$

$$\underline{157506622}$$

$$\underline{\underline{151215}}$$

Step 2: Factorization

	1	2	3	4	5	6	7
2	1	0	0	0	0	0	0
3	1	0	0	0	0	0	0
4	0	1	1	0	0	0	0
5	0	1	1	1	0	0	0
6	1	1	0	0	0	0	0
7	1	0	1	1	1	0	0
8	1	1	0	1	1	1	0
9	1	1	1	1	0	1	0
10	1	0	0	0	1	1	1
11	1	0	1	0	0	1	0

	1	2	3	4	5	6	7
2	1	0	0	0	0	0	0
3	1	0	0	0	0	0	0
4	0	1	1	0	0	0	0
5	0	1	1	1	0	0	0
6	1	1	0	0	0	0	0
7	1	0	1	1	1	0	0
8	1	1	0	1	1	1	0
9	1	1	1	1	0	1	0
10	1	0	0	0	1	1	1
11	1	0	1	0	0	1	0

Select

	1	4	6	7
2	1	0	0	0
3	1	0	0	0
4	0	0	0	0
5	0	1	0	0
6	1	0	0	0
7	1	1	0	0
8	1	1	1	0
9	1	1	1	0
10	1	0	1	1
11	1	0	1	0

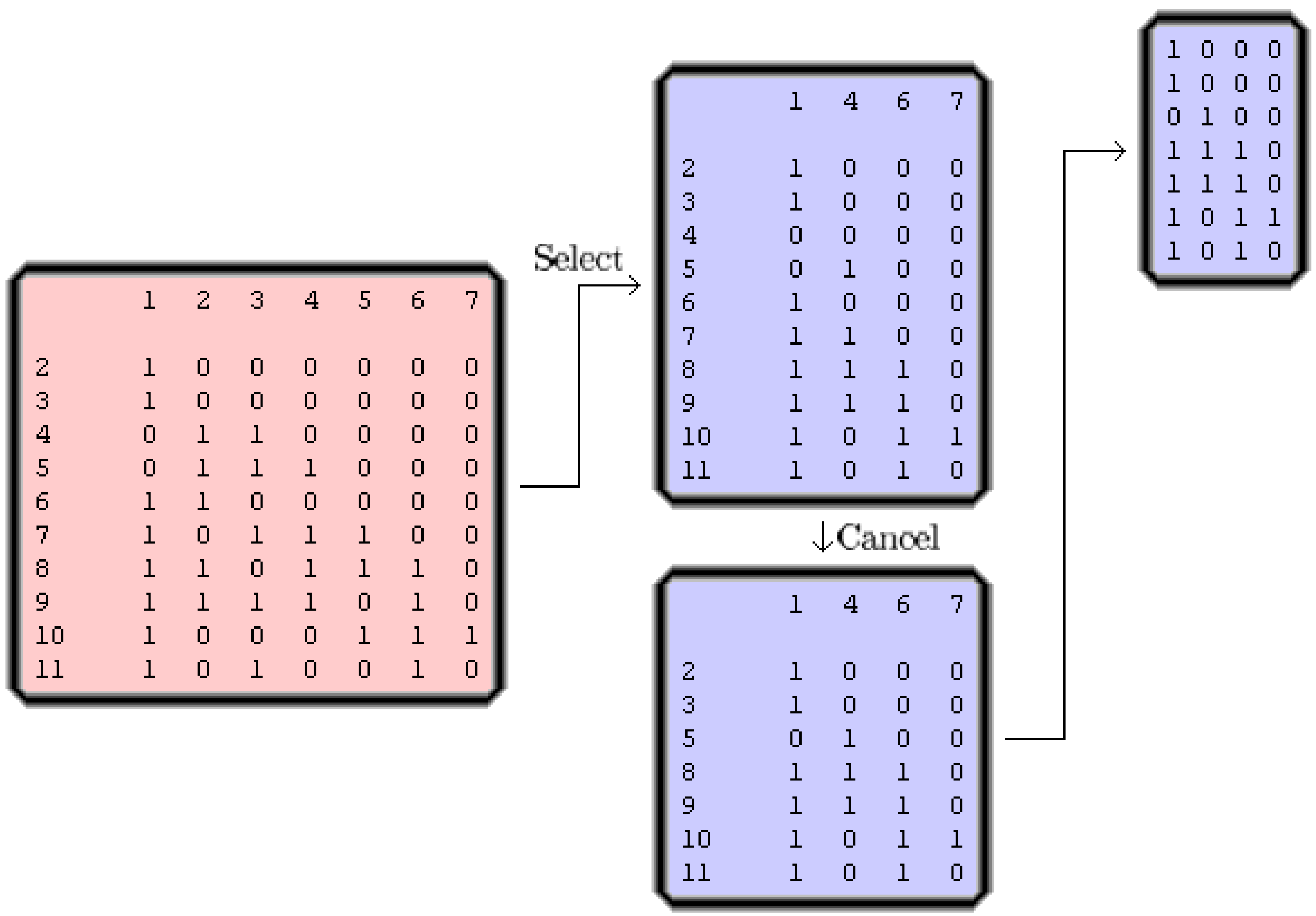
	1	2	3	4	5	6	7
2	1	0	0	0	0	0	0
3	1	0	0	0	0	0	0
4	0	1	1	0	0	0	0
5	0	1	1	1	0	0	0
6	1	1	0	0	0	0	0
7	1	0	1	1	1	0	0
8	1	1	0	1	1	1	0
9	1	1	1	1	0	1	0
10	1	0	0	0	1	1	1
11	1	0	1	0	0	1	0

Select →

	1	4	6	7
2	1	0	0	0
3	1	0	0	0
4	0	0	0	0
5	0	1	0	0
6	1	0	0	0
7	1	1	0	0
8	1	1	1	0
9	1	1	1	0
10	1	0	1	1
11	1	0	1	0

↓Cancel

	1	4	6	7
2	1	0	0	0
3	1	0	0	0
5	0	1	0	0
8	1	1	1	0
9	1	1	1	0
10	1	0	1	1
11	1	0	1	0


1 2 3 4 5 6 7
2 1 0 0 0 0 0 0
3 1 0 0 0 0 0 0
4 0 1 1 0 0 0 0
5 0 1 1 1 0 0 0
6 1 1 0 0 0 0 0
7 1 0 1 1 1 0 0
8 1 1 0 1 1 1 0
9 1 1 1 1 0 1 0
10 1 0 0 0 1 1 1
11 1 0 1 0 0 1 0
Select
1 4 6 7
2 1 0 0 0
3 1 0 0 0
4 0 0 0 0
5 0 1 0 0
6 1 0 0 0
7 1 1 0 0
8 1 1 1 0
9 1 1 1 0
10 1 0 1 1
11 1 0 1 0
↓Cancel
1 4 6 7
2 1 0 0 0
3 1 0 0 0
5 0 1 0 0
8 1 1 1 0
9 1 1 1 0
10 1 0 1 1
11 1 0 1 0
1 0 0 0
1 0 0 0
0 1 0 0
1 1 1 0
1 1 1 0
1 0 1 1
1 0 1 0

	1	2	3	4	5	6	7
2	1	0	0	0	0	0	0
3	1	0	0	0	0	0	0
4	0	1	1	0	0	0	0
5	0	1	1	1	0	0	0
6	1	1	0	0	0	0	0
7	1	0	1	1	1	0	0
8	1	1	0	1	1	1	0
9	1	1	1	1	0	1	0
10	1	0	0	0	1	1	1
11	1	0	1	0	0	1	0

Select

	1	4	6	7
2	1	0	0	0
3	1	0	0	0
4	0	0	0	0
5	0	1	0	0
6	1	0	0	0
7	1	1	0	0
8	1	1	1	0
9	1	1	1	0
10	1	0	1	1
11	1	0	1	0

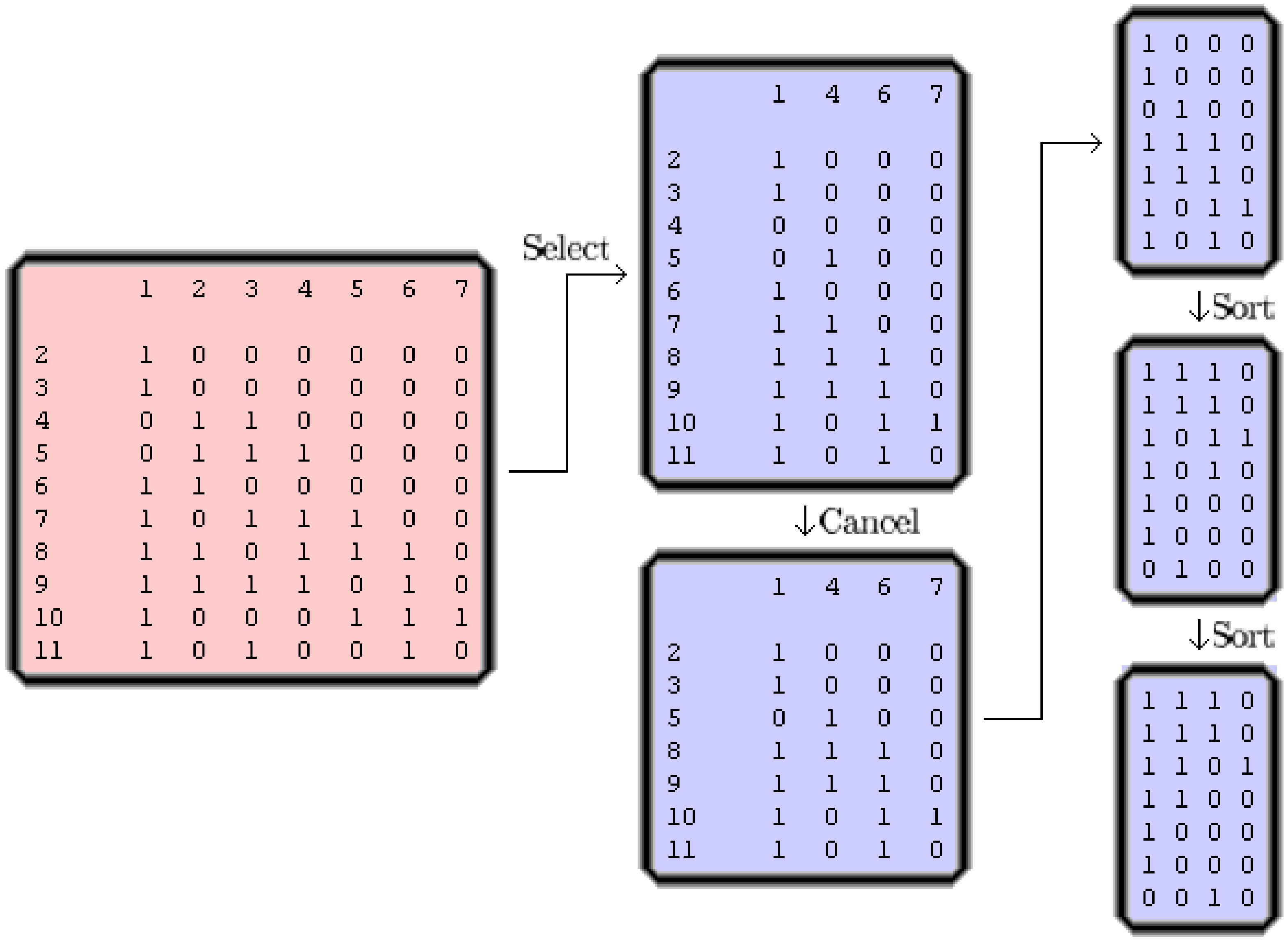
↓Cancel

	1	4	6	7
2	1	0	0	0
3	1	0	0	0
5	0	1	0	0
8	1	1	1	0
9	1	1	1	0
10	1	0	1	1
11	1	0	1	0

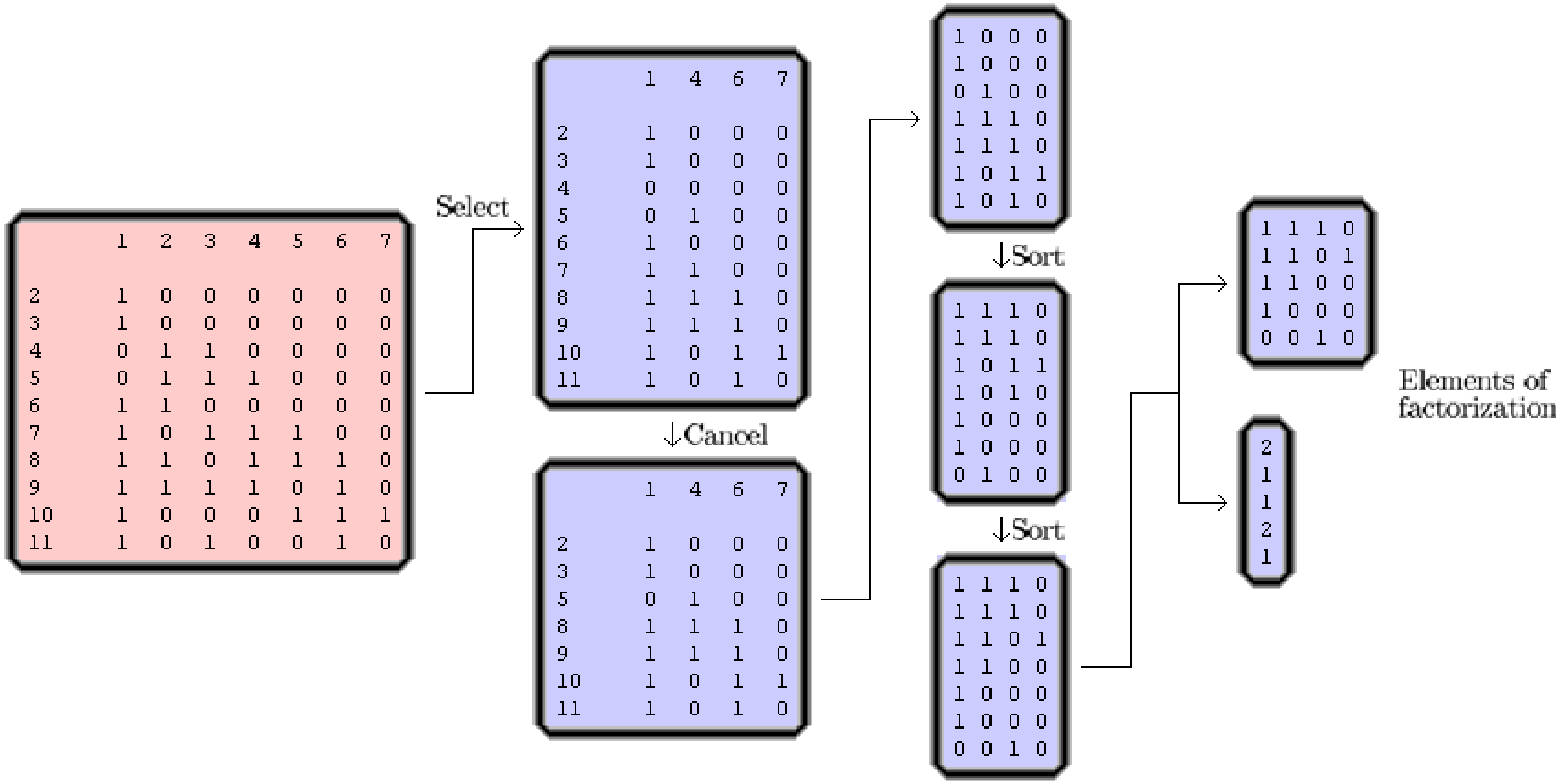
1	0	0	0
1	0	0	0
0	1	0	0
1	1	1	0
1	1	1	0
1	0	1	1
1	0	1	0

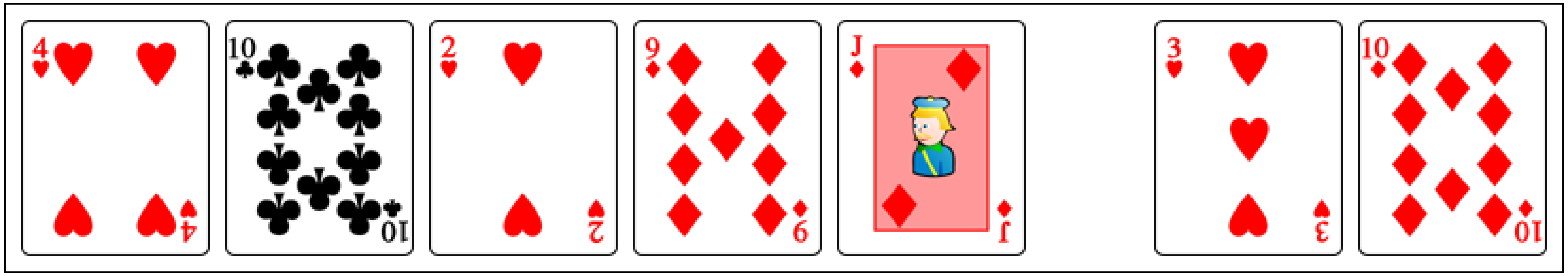
↓Sort

1	1	1	0
1	1	1	0
1	0	1	1
1	0	1	0
1	0	0	0
1	0	0	0
0	1	0	0



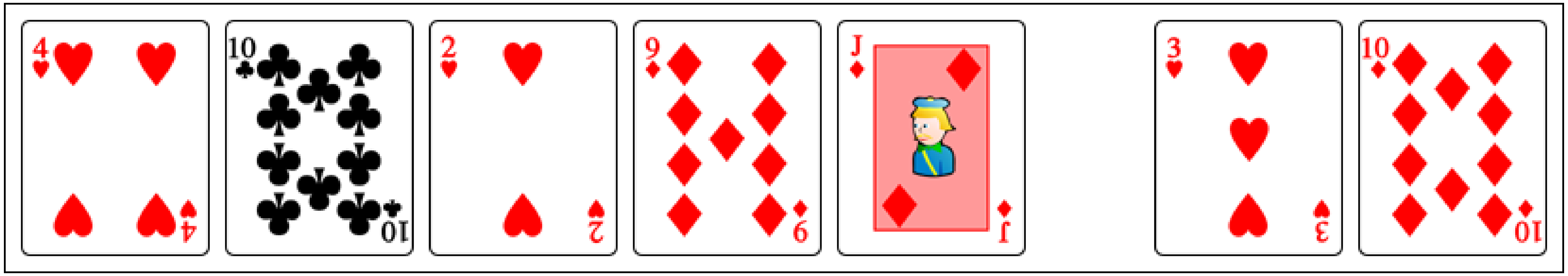
1 2 3 4 5 6 7
2 1 0 0 0 0 0 0
3 1 0 0 0 0 0 0
4 0 1 1 0 0 0 0
5 0 1 1 1 0 0 0
6 1 1 0 0 0 0 0
7 1 0 1 1 1 0 0
8 1 1 0 1 1 1 0
9 1 1 1 1 0 1 0
10 1 0 0 0 1 1 1
11 1 0 1 0 0 1 0
Select
1 4 6 7
2 1 0 0 0
3 1 0 0 0
4 0 0 0 0
5 0 1 0 0
6 1 0 0 0
7 1 1 0 0
8 1 1 1 0
9 1 1 1 0
10 1 0 1 1
11 1 0 1 0
↓Cancel
1 4 6 7
2 1 0 0 0
3 1 0 0 0
5 0 1 0 0
8 1 1 1 0
9 1 1 1 0
10 1 0 1 1
11 1 0 1 0
1 0 0 0
1 0 0 0
0 1 0 0
1 1 1 0
1 1 1 0
1 0 1 1
1 0 1 0
↓Sort
1 1 1 0
1 1 1 0
1 0 1 1
1 0 1 0
1 0 0 0
1 0 0 0
0 1 0 0
↓Sort
1 1 1 0
1 1 1 0
1 1 0 1
1 1 0 0
1 0 0 0
1 0 0 0
0 0 1 0


Select
↓Cancel
↓Sort
↓Sort
Elements of factorization



151215 Graphs

62593054397 Matrices

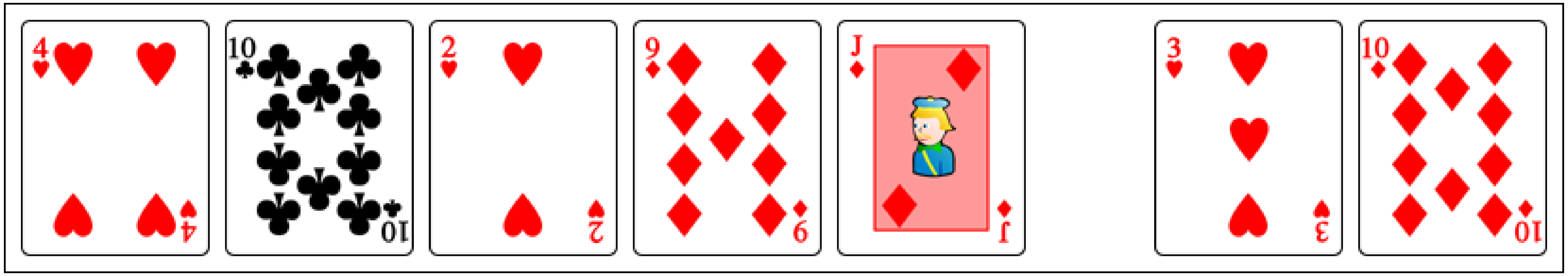


151215 Graphs

62593054397 Matrices

Reduce to:

333619695 Matrices



151215 Graphs

62593054397 Matrices

Reduce to:

333619695 Matrices

A factor of: 188

Step 3: Evaluation of matrices

$$\sum_{(i_1,\ldots,i_n)\in\{1,\ldots,m'\}^n} \left(\prod_{j=1}^{n} a'_{i_j,j} \left(b_{i_j} - \#\left\{j' \in \{1,\ldots,j-1\} \,|\, i_{j'} = i_j\right\}\right)\right)$$

Ways to select one "1" in every column of A such that no two "1"s share a row

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\sum_{(i_1,\ldots,i_n)\in\{1,\ldots,m'\}^n} \left(\prod_{j=1}^{n} a'_{i_j,j} \left(b_{i_j} - \#\left\{j' \in \{1,\ldots,j-1\} \,|\, i_{j'} = i_j\right\}\right)\right)$$

Ways to select one "1" in every column of A such that no two "1"'s share a row

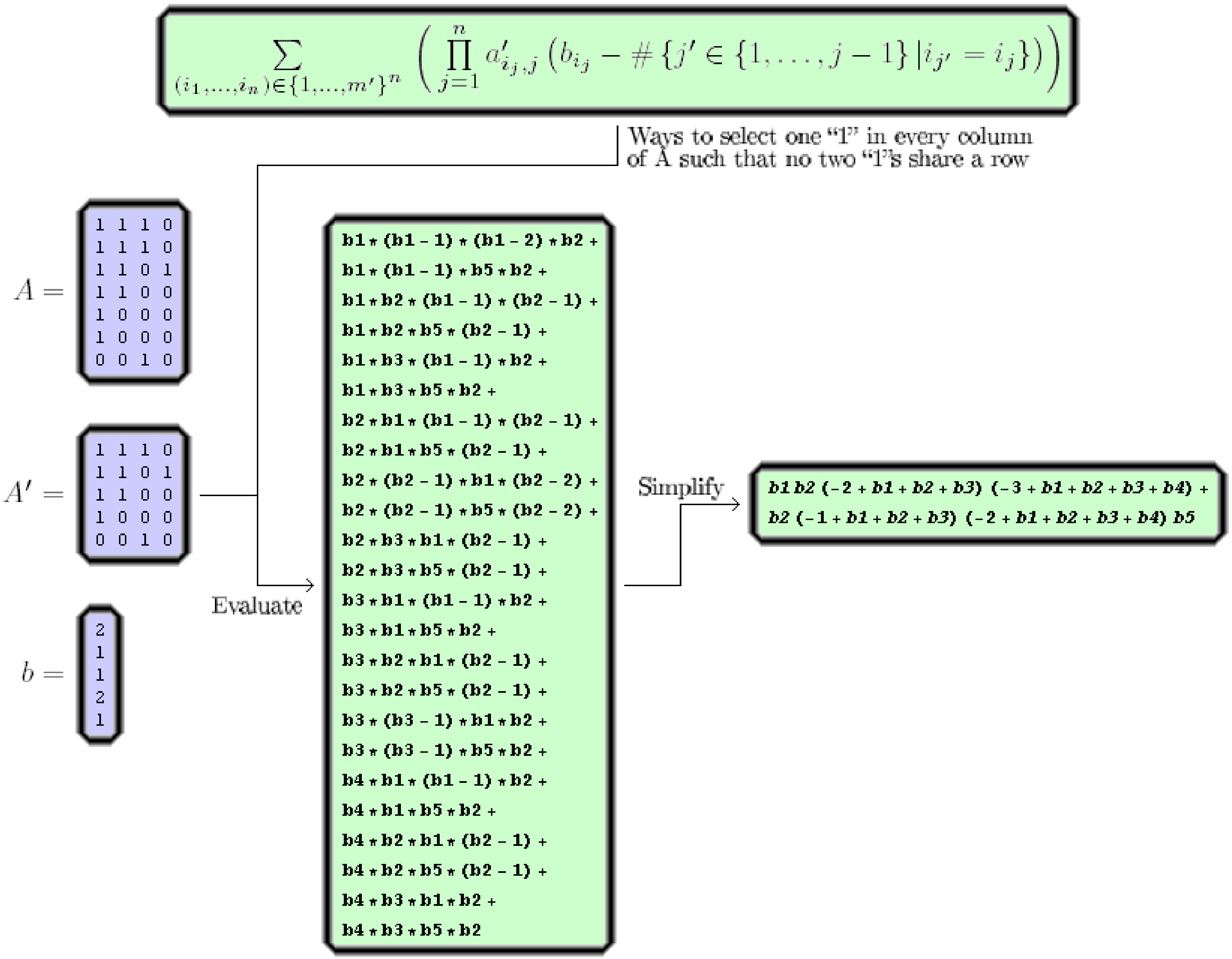
$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

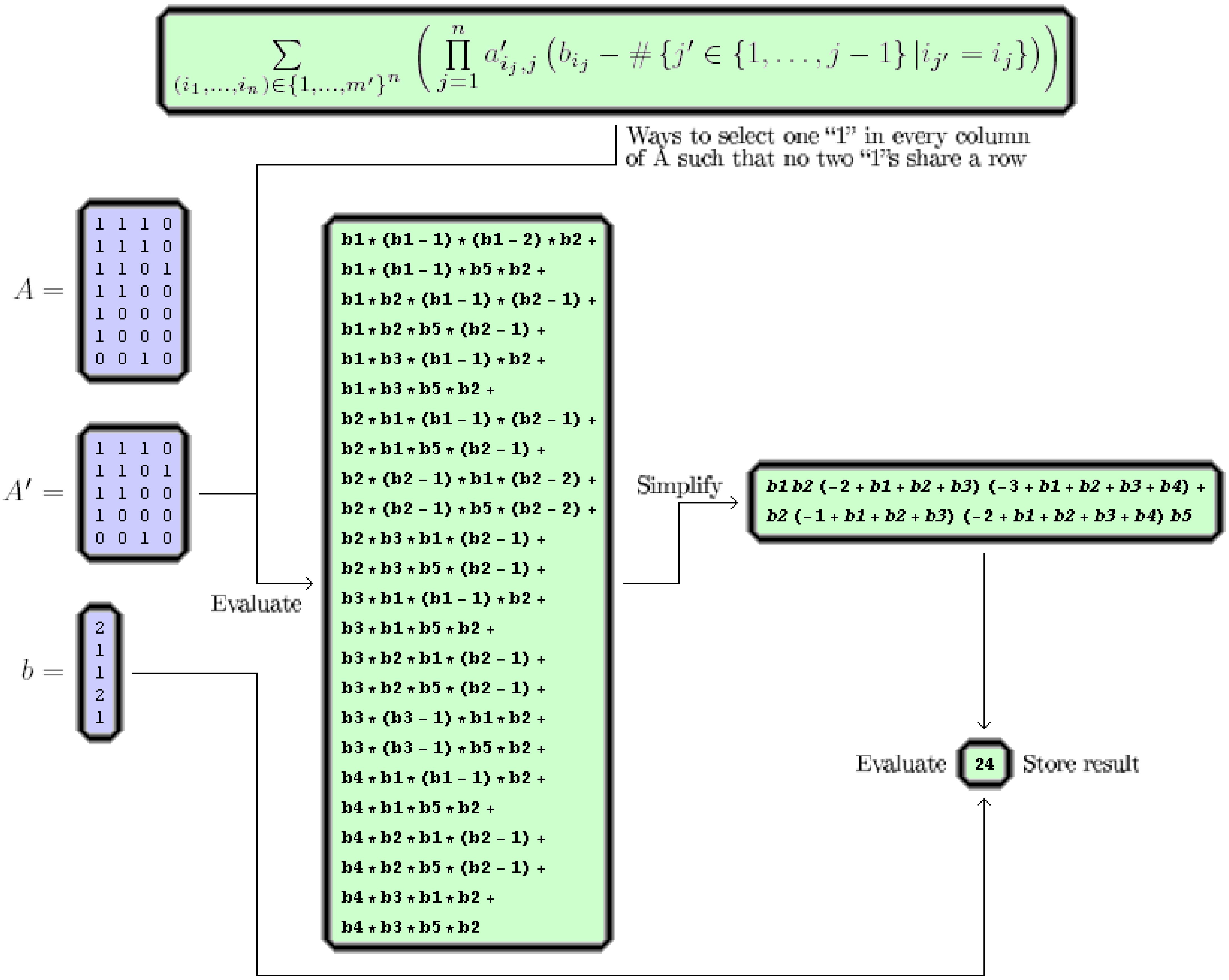
$$b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

Evaluate

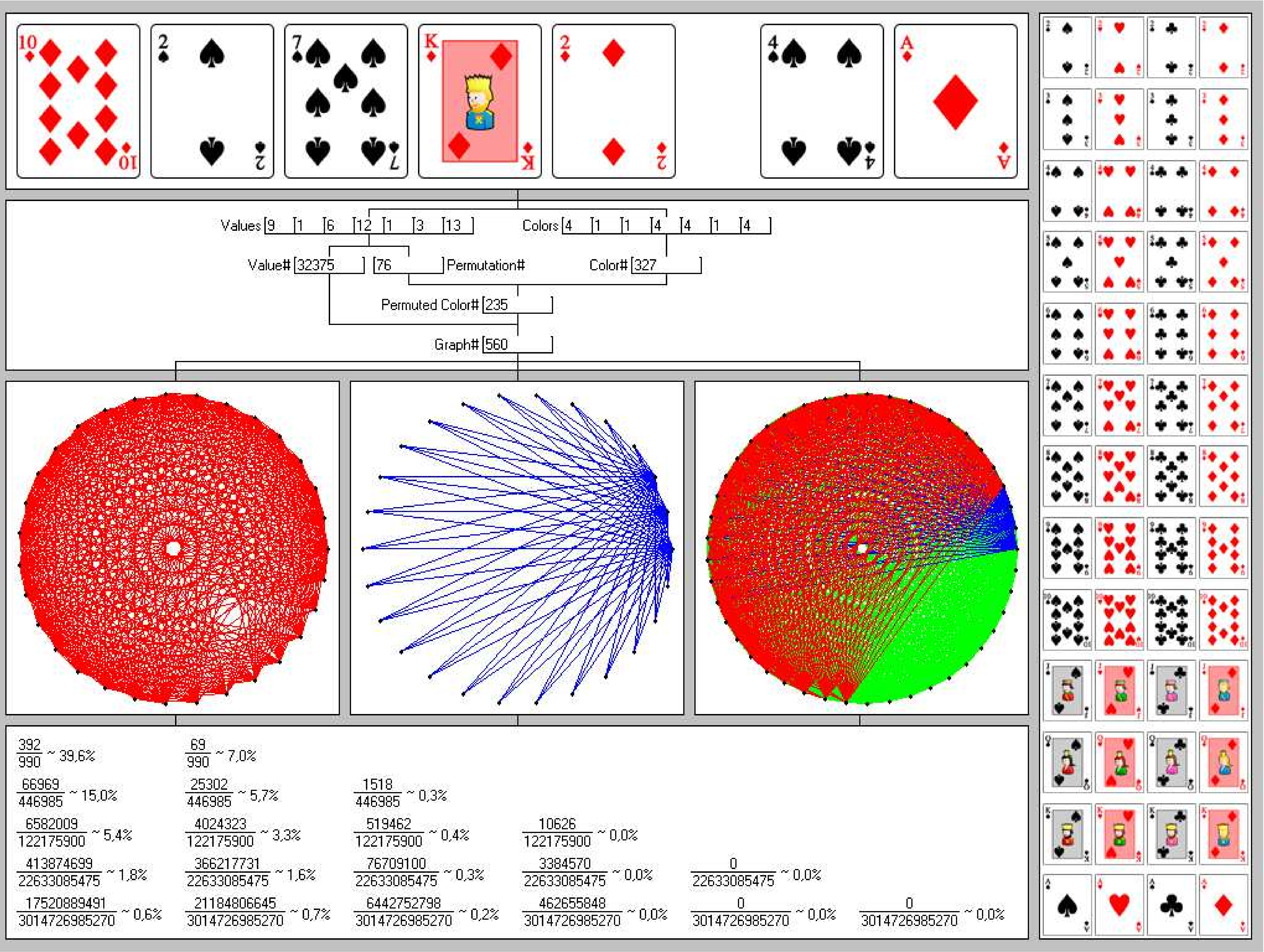
```
b1 * (b1 - 1) * (b1 - 2) * b2 +
b1 * (b1 - 1) * b5 * b2 +
b1 * b2 * (b1 - 1) * (b2 - 1) +
b1 * b2 * b5 * (b2 - 1) +
b1 * b3 * (b1 - 1) * b2 +
b1 * b3 * b5 * b2 +
b2 * b1 * (b1 - 1) * (b2 - 1) +
b2 * b1 * b5 * (b2 - 1) +
b2 * (b2 - 1) * b1 * (b2 - 2) +
b2 * (b2 - 1) * b5 * (b2 - 2) +
b2 * b3 * b1 * (b2 - 1) +
b2 * b3 * b5 * (b2 - 1) +
b3 * b1 * (b1 - 1) * b2 +
b3 * b1 * b5 * b2 +
b3 * b2 * b1 * (b2 - 1) +
b3 * b2 * b5 * (b2 - 1) +
b3 * (b3 - 1) * b1 * b2 +
b3 * (b3 - 1) * b5 * b2 +
b4 * b1 * (b1 - 1) * b2 +
b4 * b1 * b5 * b2 +
b4 * b2 * b1 * (b2 - 1) +
b4 * b2 * b5 * (b2 - 1) +
b4 * b3 * b1 * b2 +
b4 * b3 * b5 * b2
```



$\sum_{(i_1,\ldots,i_n)\in\{1,\ldots,m'\}^n} \left(\prod_{j=1}^{n} a'_{i_j,j} \left(b_{i_j} - \#\{j' \in \{1,\ldots,j-1\} \mid i_{j'} = i_j\}\right)\right)$
Ways to select one "1" in every column of A such that no two "1"'s share a row
A =
1 1 1 0
1 1 1 0
1 1 0 1
1 1 0 0
1 0 0 0
1 0 0 0
0 0 1 0
A′ =
1 1 1 0
1 1 0 1
1 1 0 0
1 0 0 0
0 0 1 0
b =
2
1
1
2
1
Evaluate
b1 * (b1 - 1) * (b1 - 2) * b2 +
b1 * (b1 - 1) * b5 * b2 +
b1 * b2 * (b1 - 1) * (b2 - 1) +
b1 * b2 * b5 * (b2 - 1) +
b1 * b3 * (b1 - 1) * b2 +
b1 * b3 * b5 * b2 +
b2 * b1 * (b1 - 1) * (b2 - 1) +
b2 * b1 * b5 * (b2 - 1) +
b2 * (b2 - 1) * b1 * (b2 - 2) +
b2 * (b2 - 1) * b5 * (b2 - 2) +
b2 * b3 * b1 * (b2 - 1) +
b2 * b3 * b5 * (b2 - 1) +
b3 * b1 * (b1 - 1) * b2 +
b3 * b1 * b5 * b2 +
b3 * b2 * b1 * (b2 - 1) +
b3 * b2 * b5 * (b2 - 1) +
b3 * (b3 - 1) * b1 * b2 +
b3 * (b3 - 1) * b5 * b2 +
b4 * b1 * (b1 - 1) * b2 +
b4 * b1 * b5 * b2 +
b4 * b2 * b1 * (b2 - 1) +
b4 * b2 * b5 * (b2 - 1) +
b4 * b3 * b1 * b2 +
b4 * b3 * b5 * b2
Simplify
b1 b2 (-2 + b1 + b2 + b3) (-3 + b1 + b2 + b3 + b4) +
b2 (-1 + b1 + b2 + b3) (-2 + b1 + b2 + b3 + b4) b5


$$\sum_{(i_1,\ldots,i_n)\in\{1,\ldots,m'\}^n}\left(\prod_{j=1}^{n} a'_{i_j,j}\left(b_{i_j}-\#\{j'\in\{1,\ldots,j-1\}\,|\,i_{j'}=i_j\}\right)\right)$$
Ways to select one "1" in every column of A such that no two "1"'s share a row
$A =$
1 1 1 0
1 1 1 0
1 1 0 1
1 1 0 0
1 0 0 0
1 0 0 0
0 0 1 0
$A' =$
1 1 1 0
1 1 0 1
1 1 0 0
1 0 0 0
0 0 1 0
$b =$
2
1
1
2
1
Evaluate
b1 * (b1 - 1) * (b1 - 2) * b2 +
b1 * (b1 - 1) * b5 * b2 +
b1 * b2 * (b1 - 1) * (b2 - 1) +
b1 * b2 * b5 * (b2 - 1) +
b1 * b3 * (b1 - 1) * b2 +
b1 * b3 * b5 * b2 +
b2 * b1 * (b1 - 1) * (b2 - 1) +
b2 * b1 * b5 * (b2 - 1) +
b2 * (b2 - 1) * b1 * (b2 - 2) +
b2 * (b2 - 1) * b5 * (b2 - 2) +
b2 * b3 * b1 * (b2 - 1) +
b2 * b3 * b5 * (b2 - 1) +
b3 * b1 * (b1 - 1) * b2 +
b3 * b1 * b5 * b2 +
b3 * b2 * b1 * (b2 - 1) +
b3 * b2 * b5 * (b2 - 1) +
b3 * (b3 - 1) * b1 * b2 +
b3 * (b3 - 1) * b5 * b2 +
b4 * b1 * (b1 - 1) * b2 +
b4 * b1 * b5 * b2 +
b4 * b2 * b1 * (b2 - 1) +
b4 * b2 * b5 * (b2 - 1) +
b4 * b3 * b1 * b2 +
b4 * b3 * b5 * b2
Simplify
b1 b2 (-2 + b1 + b2 + b3) (-3 + b1 + b2 + b3 + b4) +
b2 (-1 + b1 + b2 + b3) (-2 + b1 + b2 + b3 + b4) b5
Evaluate
24
Store result

Step 4: Computation of coefficients


Values 9 1 6 12 1 3 13
Colors 4 1 1 4 4 1 4
Value# 32375
76
Permutation#
Color# 327
Permuted Color# 235
Graph# 560
392/990 ~ 39,6%
69/990 ~ 7,0%
66969/446985 ~ 15,0%
25302/446985 ~ 5,7%
1518/446985 ~ 0,3%
6582009/122175900 ~ 5,4%
4024323/122175900 ~ 3,3%
519462/122175900 ~ 0,4%
10626/122175900 ~ 0,0%
413874699/22633085475 ~ 1,8%
366217731/22633085475 ~ 1,6%
76709100/22633085475 ~ 0,3%
3384570/22633085475 ~ 0,0%
0/22633085475 ~ 0,0%
17520889491/3014726985270 ~ 0,6%
21184806645/3014726985270 ~ 0,7%
6442752798/3014726985270 ~ 0,2%
462655848/3014726985270 ~ 0,0%
0/3014726985270 ~ 0,0%
0/3014726985270 ~ 0,0%