An introduction to p-adic systems: A new kind of number system

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A crash course in p-adic numbers

p-adic numbers: the good/evil/equally likeable (?) twin of real numbers

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Real numbers $\mathbb{R}$	10-adic numbers $\mathbb{Q}_{10}$

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- $\{n/d \in \mathbb{Q} \mid \gcd(d, 10) = 1\} \subseteq \mathbb{Z}_{10}$: digits ultimately periodic ($3/7 = \overline{857142}9$)

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- The modulo function generalizes to $\mathbb{Z}_{10}$ for powers of 10: $\ldots 357142 \,\%\, 10^3 = 142$

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More facts on p-adic integers

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\vdots \\
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A crash course in p-adic numbers

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More examples of number systems on $\mathbb{Z}_2$ and $\mathbb{Z}_3$

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$T_2 = (x, x)$ "=" $(x, x-1)$

What is a p-adic system?

In short: A p-adic system is a number system on $\mathbb{Z}_p$ which respects $\|\cdot\|_p$

first k digits of expansions of m and n coincide $\Leftrightarrow$ $m \equiv n \mod p^k$

Formal: Let $F = (F[0], \ldots, F[p-1])$ with $F[r] : \mathbb{Z}_p \to \mathbb{Z}_p$ and define

$$F : \mathbb{Z}_p \to \mathbb{Z}_p$$
$$n \mapsto \frac{F[n \% p](n) - \cancel{F[n \% p](n) \% p}}{p}$$

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Examples: $T_{10} = (x,x,x,x,x,x,x,x,x,x)$ | $(x,\ldots,x)(153) = 53$

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Examples:

$T_{10} = (x, x, x, x, x, x, x, x, x, x)$		$(x, \ldots, x)(153) = 53$
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Examples:

T_{10}	$= (x, x, x, x, x, x, x, x, x, x)$	$(x, \ldots, x)(153) = 53$
T_2	$= (x, x)$ "=" $(x, x-1)$	$(x, x)(17) = 8$
T_C	$= (x, 3x+1)$	$(x, 3x+1)(9) = 14$

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Examples:

T_{10}	$= (x, x, x, x, x, x, x, x, x, x)$	$(x, \ldots, x)(153) = 53$
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Examples:

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$T_{a,b}$	$= (x, x-a, x-b)$	$(x, x-1, x+1)(5) = 2$

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F-sequence of n : $S(F)[n] = (F^k(n))_{k \in \mathbb{N}_0}$

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Examples: $S(T_C)[17] = (17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, \ldots)$

In short: A p-adic system is a number system on $\mathbb{Z}_p$ which respects $\|\cdot\|_p$

first k digits of expansions of m and n coincide $\Leftrightarrow$ $m \equiv n \mod p^k$

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Examples:

T_{10}	$= (x,x,x,x,x,x,x,x,x,x)$	$(x,\ldots,x)(153) = 53$
T_2	$= (x,x)$ "$=$" $(x, x-1)$	$(x,x)(17) = 8$
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F-(digit-)expansion of n: $D(F)[n] = (F^k(n) \,\%\, p)_{k \in \mathbb{N}_0}$

Examples: $S(T_C)[17] = (17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, \ldots)$

$D(T_C)[17] = (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, \ldots)$

In short: A p-adic system is a number system on $\mathbb{Z}_p$ which respects $\|\cdot\|_p$

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Examples: $S(T_C)[17] = (17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, \ldots)$

$D(T_C)[17] = (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, \ldots)$

F is called a p-adic system if for all $m, n \in \mathbb{Z}_p$ and $k \in \mathbb{N}$

$D(F)[m][0, k-1] = D(F)[n][0, k-1] \Leftrightarrow m \equiv n \mod p^k$

What is a p-adic system?

$F = (F[0], \ldots, F[p-1]) : \mathbb{Z}_p \to \mathbb{Z}_p, \quad n \mapsto \left(F[n \,\%\, p](n) - \cancel{F[n \,\%\, p](n) \,\%\, p}\right)/p$

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Examples: $T_n = (x, \ldots, x)$, $T_C = (x, 3x+1)$, $T_{a,b} = (x, x-a, x-b)$

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⋮	⋮	⋮	⋮	⋮	⋰
1	1	2	1	2	⋯
2	2	1	2	1	⋯
3	3	5	8	4	⋯
4	4	2	1	2	⋯
5	5	8	4	2	⋯
6	6	3	5	8	⋯
7	7	11	17	26	⋯
8	8	4	2	1	⋯
9	9	14	7	11	⋯
10	10	5	8	4	⋯
11	11	17	26	13	⋯
12	12	6	3	5	⋯
13	13	20	10	5	⋯
14	14	7	11	17	⋯
15	15	23	35	53	⋯
16	16	8	4	2	⋯
⋮	⋮	⋮	⋮	⋮	⋱
$S(T_C)$	0	1	2	3	⋯

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	2	1	2	$\cdots$
2	2	1	2	1	$\cdots$
3	3	5	8	4	$\cdots$
4	4	2	1	2	$\cdots$
5	5	8	4	2	$\cdots$
6	6	3	5	8	$\cdots$
7	7	11	17	26	$\cdots$
8	8	4	2	1	$\cdots$
9	9	14	7	11	$\cdots$
10	10	5	8	4	$\cdots$
11	11	17	26	13	$\cdots$
12	12	6	3	5	$\cdots$
13	13	20	10	5	$\cdots$
14	14	7	11	17	$\cdots$
15	15	23	35	53	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$S(T_C)$	0	1	2	3	$\cdots$

$S(F) \in \left(\mathbb{Z}_p^{\mathbb{N}_0} \right)^{\mathbb{Z}_p}$

F-sequence table

$F = (F[0], \ldots, F[p-1]) : \mathbb{Z}_p \to \mathbb{Z}_p, \quad n \mapsto \Big(F[n \,\%\, p](n) - \cancel{F[n \,\%\, p](n) \,\%\, p}\Big)/p$

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$S(T_C)$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	2	1	2	$\cdots$
2	2	1	2	1	$\cdots$
3	3	5	8	4	$\cdots$
4	4	2	1	2	$\cdots$
5	5	8	4	2	$\cdots$
6	6	3	5	8	$\cdots$
7	7	11	17	26	$\cdots$
8	8	4	2	1	$\cdots$
9	9	14	7	11	$\cdots$
10	10	5	8	4	$\cdots$
11	11	17	26	13	$\cdots$
12	12	6	3	5	$\cdots$
13	13	20	10	5	$\cdots$
14	14	7	11	17	$\cdots$
15	15	23	35	53	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$D(T_C)$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	0	1	0	$\cdots$
2	0	1	0	1	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	0	0	$\cdots$
6	0	1	1	0	$\cdots$
7	1	1	1	0	$\cdots$
8	0	0	0	1	$\cdots$
9	1	0	1	1	$\cdots$
10	0	1	0	0	$\cdots$
11	1	1	0	1	$\cdots$
12	0	0	1	1	$\cdots$
13	1	0	0	1	$\cdots$
14	0	1	1	1	$\cdots$
15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$S(F) \in \left(\mathbb{Z}_p{}^{\mathbb{N}_0}\right)^{\mathbb{Z}_p}$

F-sequence table

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$S(T_C)$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	2	1	2	$\cdots$
2	2	1	2	1	$\cdots$
3	3	5	8	4	$\cdots$
4	4	2	1	2	$\cdots$
5	5	8	4	2	$\cdots$
6	6	3	5	8	$\cdots$
7	7	11	17	26	$\cdots$
8	8	4	2	1	$\cdots$
9	9	14	7	11	$\cdots$
10	10	5	8	4	$\cdots$
11	11	17	26	13	$\cdots$
12	12	6	3	5	$\cdots$
13	13	20	10	5	$\cdots$
14	14	7	11	17	$\cdots$
15	15	23	35	53	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$S(F) \in \left({\mathbb{Z}_p}^{\mathbb{N}_0}\right)^{\mathbb{Z}_p}$

F-sequence table

$D(T_C)$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	0	1	0	$\cdots$
2	0	1	0	1	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	0	0	$\cdots$
6	0	1	1	0	$\cdots$
7	1	1	1	0	$\cdots$
8	0	0	0	1	$\cdots$
9	1	0	1	1	$\cdots$
10	0	1	0	0	$\cdots$
11	1	1	0	1	$\cdots$
12	0	0	1	1	$\cdots$
13	1	0	0	1	$\cdots$
14	0	1	1	1	$\cdots$
15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$D(F) \in \left(\{0, \dots, p-1\}^{\mathbb{N}_0}\right)^{\mathbb{Z}_p}$

F-digit table

What is a p-adic system?

$F = (F[0], \dots, F[p-1]) : \mathbb{Z}_p \to \mathbb{Z}_p, \quad n \mapsto \left(F[n \,\%\, p](n) - \cancel{F[n \,\%\, p](n) \,\%\, p}\right)/p$

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$S(T_C)$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$⋰$
1	1	2	1	2	$\cdots$
2	2	1	2	1	$\cdots$
3	3	5	8	4	$\cdots$
4	4	2	1	2	$\cdots$
5	5	8	4	2	$\cdots$
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8	8	4	2	1	$\cdots$
9	9	14	7	11	$\cdots$
10	10	5	8	4	$\cdots$
11	11	17	26	13	$\cdots$
12	12	6	3	5	$\cdots$
13	13	20	10	5	$\cdots$
14	14	7	11	17	$\cdots$
15	15	23	35	53	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$S(F) \in \left(\mathbb{Z}_p^{\mathbb{N}_0}\right)^{\mathbb{Z}_p}$

F-sequence table

$D(T_C)$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$⋰$
1	1	0	1	0	$\cdots$
2	0	1	0	1	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	0	0	$\cdots$
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13	1	0	0	1	$\cdots$
14	0	1	1	1	$\cdots$
15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$D(F) \in \left(\{0, \dots, p-1\}^{\mathbb{N}_0}\right)^{\mathbb{Z}_p}$

F-digit table

$F = (F[0], \ldots, F[p-1]) : \mathbb{Z}_p \to \mathbb{Z}_p, \quad n \mapsto \left(F[n \,\%\, p](n) - \cancel{F[n \,\%\, p](n) \,\%\, p}\right)/p$

F is called a p-adic system if $D(F)[m][0, k-1] = D(F)[n][0, k-1] \Leftrightarrow m \equiv n \mod p^k$

Properties of p-adic systems

- $D(F)$ defines a bijection between $\mathbb{Z}_p$ and $\{0, \ldots, p-1\}^{\mathbb{N}_0}$:

$$F = (F[0], \ldots, F[p-1]) : \mathbb{Z}_p \to \mathbb{Z}_p, \quad n \mapsto \left(F[n \,\%\, p](n) - \cancel{F[n \,\%\, p](n) \,\%\, p}\right)/p$$

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Properties of p-adic systems

- $D(F)$ defines a bijection between $\mathbb{Z}_p$ and $\{0, \ldots, p-1\}^{\mathbb{N}_0}$:
  the expansions of all p-adic integers are unique and every possible expansion occurs

$$F = (F[0], \dots, F[p-1]) : \mathbb{Z}_p \to \mathbb{Z}_p, \quad n \mapsto \Big(F[n \,\%\, p](n) - \cancel{F[n \,\%\, p](n) \,\%\, p}\Big)/p$$

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  One can always make $F[r]$ (p,r)-suitable by only changing a_1

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More examples

- $T_n = (x, \ldots, x)$, $T_C = (x, 3x+1)$, $T_{a,b} = (x, x-a, x-b)$

$S(T_C)$	0	1	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	2	1	$\cdots$
2	2	1	2	$\cdots$
3	3	5	8	$\cdots$
4	4	2	1	$\cdots$
5	5	8	4	$\cdots$
6	6	3	5	$\cdots$
7	7	11	17	$\cdots$
8	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$D(T_C)$	0	1	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	0	1	$\cdots$
2	0	1	0	$\cdots$
3	1	1	0	$\cdots$
4	0	0	1	$\cdots$
5	1	0	0	$\cdots$
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- $(7x^3 - 4x^2 + x - 6, 3x^7 - x + 1, x^2 + 6x + 2)$, $(\frac{32}{7}x^2 + \frac{11}{3}x - 4, \frac{13}{11}x + 5)$

⋮	⋮	⋮	⋮	⋰
1	1	34/11	64805/2541	⋯
2	2	227/21	2053/231	⋯
3	3	47/11	608/121	⋯
4	4	880/21	12621386/3087	⋯
5	5	60/11	64376/847	⋯
6	6	639/7	4346/77	⋯
7	7	73/11	777/121	⋯
8	8	3338/21	59723107/1029	⋯
⋮	⋮	⋮	⋮	⋱
$S(F)$	0	1	2	⋯

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2	0	1	1	⋯
3	1	1	0	⋯
4	0	0	0	⋯
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- $(\mathrm{i}x^2 - 6x + 5\mathrm{i}, x, x, x, x)$ where $\mathrm{i}^2 = -1$, i.e. $\mathrm{i} = \ldots 31212$ or $\mathrm{i} = \ldots 13233$

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- If $P \in \mathbb{Z}[x]$ is a p-permutation polynomial ($P : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$ bijective for all $k \in \mathbb{N}$)

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- $(7x^3 - 4x^2 + x - 6, 3x^7 - x + 1, x^2 + 6x + 2)$, $(\frac{32}{7}x^2 + \frac{11}{3}x - 4, \frac{13}{11}x + 5)$
- $(\mathrm{i}x^2 - 6x + 5\mathrm{i}, x, x, x, x)$ where $\mathrm{i}^2 = -1$, i.e. $\mathrm{i} = \ldots 31212$ or $\mathrm{i} = \ldots 13233$
- If $P \in \mathbb{Z}[x]$ is a p-permutation polynomial ($P : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$ bijective for all $k \in \mathbb{N}$) and $P : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is the identity function,

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  Example: $P(x) = 10x^2 - 3x + 4$ is a 2-permutation polynomial

P	
⋮	⋮
1	11
2	38
3	85
4	152
5	239
6	346
7	473
8	620
⋮	⋮

$D(P)$	0	1	2	$\cdots$
⋮	⋮	⋮	⋮	⋰
1	1	1	0	$\cdots$
2	0	1	1	$\cdots$
3	1	0	1	$\cdots$
4	0	0	0	$\cdots$
5	1	1	1	$\cdots$
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Corollary: If $P \in \mathbb{Z}_p[x]$ with $P(r) \equiv 0 \mod p$ and $\gcd(p, P'(r)) = 1$,
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Obligatory proof

Lemma: If $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is (p, r)-suitable, then so is $g : \mathbb{Z}_p \to \mathbb{Z}_p$, $x \mapsto f(x) + px$

Remember: f (p, r)-suitable $\overset{\text{Def}}{\Leftrightarrow} (f(m) \equiv f(n) \mod p^k \Leftrightarrow m \equiv n \mod p^k)$

$\Leftrightarrow (x, \ldots, x, \underbrace{f(x)}_{r\text{-th position}}, x, \ldots, x)$ is a p-adic system

Theorem: If $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is (p, r)-suitable and $f(n) \equiv 0 \mod p$ for all $n \equiv r \mod p$,
then f has a unique root $z \in \mathbb{Z}_p$ with $z \equiv r \mod p$

Proof: Let $F := (x, \ldots, x, f(x) + px, x, \ldots, x)$
F is a p-adic system
There is a unique $z \in \mathbb{Z}_p$ with $z \equiv r \mod p$ such that $D(F)[z] = (r, r, r, \ldots)$.
Note: $S(F)[n]$ ultimately periodic $\Leftrightarrow D(F)[n]$ ultimately periodic
lengths of initial parts and periods are equal
So, $z = F(z) = F[r](z)/p = (f(z) + pz)/p$, hence $f(z) = 0$ $\square$

Corollary: If $P \in \mathbb{Z}_p[x]$ with $P(r) \equiv 0 \mod p$ and $\gcd(p, P'(r)) = 1$,
then P has a unique root $z \in \mathbb{Z}_p$ with $z \equiv r \mod p$ (Hensel's Lemma!)

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1	-1
$1, 1, 0$	$-5, -7, -10$
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