DOCTORAL PROGRAM
DISCRETE MATHEMATICS
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Project 06

Shift Radix Systems (SRS) with Finiteness Property

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Introduction

For $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ the almost linear mapping

$$\tau_\mathbf{r} : \mathbb{Z}^d \mapsto \mathbb{Z}^d$$
$$\mathbf{x} = (x_1, \dots, x_d) \to (x_2, \dots, x_d, -\lfloor \mathbf{rx} \rfloor)$$

is called the d - dimensional SRS associated with $\mathbf{r}$.

$$\mathcal{D}_d \;:= \{\mathbf{r} \in \mathbb{R}^d \mid \text{orbits of } \tau_\mathbf{r} \text{ ultimately periodic}\}$$
$$\mathcal{D}_d^{(0)} := \{\mathbf{r} \in \mathbb{R}^d \mid \text{orbits of } \tau_\mathbf{r} \text{ end up in } \mathbf{0}\}$$

Elements of $\mathcal{D}_d^{(0)}$ are said to have the finiteness property.

Complex analogue (Gaussian SRS):
$\gamma_\mathbf{r} : \mathbb{Z}[\mathrm{i}]^d \mapsto \mathbb{Z}[\mathrm{i}]^d$ $(r \in \mathbb{C})$, $\mathcal{G}_d$, $\mathcal{G}_d^{(0)}$.

Special interest lies in the cases $d = 2$ for SRS and $d = 1$ for GSRS. We identify $\mathbb{C} \leftrightarrow \mathbb{R}^2$ and $\mathbb{Z}[\mathrm{i}] \leftrightarrow \mathbb{Z}^2$ and get

$$\tau_{(r,s)} : \mathbb{Z}^2 \mapsto \mathbb{Z}^2$$
$$(a, b) \to (b, -\lfloor ra + sb \rfloor)$$
$$\gamma_{(r,s)} : \mathbb{Z}^2 \mapsto \mathbb{Z}^2$$
$$(a, b) \to (-\lfloor ra - sb \rfloor, -\lfloor rb + sa \rfloor).$$

Main Goal and Motivation

The main objective is to characterize the sets $\mathcal{D}_d^{(0)}$ and $\mathcal{G}_d^{(0)}$. The reason is a strong relation to finiteness properties in other numeration systems.

Definition. For a non-integral, real number $\beta > 1$ the set $\mathcal{A} := \{0, 1, \dots, \lfloor \beta \rfloor\}$ is called the set of digits and for every $\gamma \in [0, \infty)$ the unique representation $\gamma = a_m\beta^m + a_{m-1}\beta^{m-1} + \cdots$, where $m \in \mathbb{Z}$, $a_i \in \mathcal{A}$, and $0 \le \gamma - \sum_{i=k}^{m} a_i\beta^i < \beta^k$ for all $k \le m$ is called the greedy expansion of γ with respect to β.
Let $\mathrm{Fin}(\beta)$ denote the set of all $\gamma \in [0, 1)$ having finite greedy expansion with respect to β. β is said to have property (F) iff $\mathrm{Fin}(\beta) = \mathbb{Z}[\frac{1}{\beta}] \cap [0, 1)$.

Remark. If β has property (F) then it is an algebraic integer.

Theorem. Let $\beta > 1$ be a non-integral algebraic integer and $(X - \beta)(X^{d-1} + r_{d-2}X^{d-2} + \cdots + r_1X + r_0)$ its minimal polynomial. Then
β has property (F) $\Leftrightarrow$ $(r_0, \dots, r_{d-2}) \in \mathcal{D}_{d-1}^{(0)}$.

(Akiyama et al. 2005)

Definition. Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0 \in \mathbb{Z}[X]$, $\mathcal{R} := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$, $\mathcal{N} := \{0, 1, \dots, |p_0| - 1\}$, and $x := X + P(X)\mathbb{Z}[X] \in \mathcal{R}$. $(P, \mathcal{N})$ is called a CNS, P a CNS polynomial and $\mathcal{N}$ the set of digits if every non-zero element $A(x) \in \mathcal{R}$ can be represented uniquely in the form $A(x) = a_mx^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0$.

Theorem. Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0 \in \mathbb{Z}[X]$. Then
P is a CNS polynomial $\Leftrightarrow$ $(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_2}{p_0}, \frac{p_1}{p_0}) \in \mathcal{D}_d^{(0)}$.

(Akiyama et al. 2005)

Main Obstacle

The main problem at characterizing $\mathcal{D}_d^{(0)}$ (and $\mathcal{G}_d^{(0)}$) are so-called critical points. It can be shown that $\mathcal{D}_d^{(0)}$ can be gained from $\mathcal{D}_d$ by subtracting certain convex polyhedra. If for a given point infinitely many of these polyhedra are necessary to characterize its neighborhood it is considered critical. In such a case the structure of $\mathcal{D}_d^{(0)}$ tends to be highly chaotic.

New Algorithms

The algorithms below compute $\mathcal{D}_d^{(0)} \cap H$ ($\mathcal{G}_d^{(0)} \cap H$) where $H \subset \mathrm{int}(\mathcal{D}_d)$ ($\mathrm{int}(\mathcal{G}_d)$) is any convex hull of finitely many points.

Algorithm. (W. [4]) Idea:

- Choose any point $\mathbf{r} \in H$ and compute the corresponding "graph of witnesses" $\mathcal{V}_\mathbf{r}$.
- Compute the polyhedron $P_\mathbf{r} := \{\mathbf{s} \in \mathbb{R}^d \mid \mathcal{V}_\mathbf{s} = \mathcal{V}_\mathbf{r}\}$ which is the solution of a system of inequalities.
- Repeat the process with a new $\mathbf{r}$ not contained in any of the previously computed $P_\mathbf{r}$ until H is completely covered.
- $\mathcal{D}_d^{(0)} \cap H$ is the union of all $P_\mathbf{r}$ for which $\mathbf{r} \in \mathcal{D}_d^{(0)}$.

Properties:

- First "real algorithm" – terminates for all inputs.
- Faster than previous methods by Brunotte.

Algorithm. (W. [4]) Idea:

- Find a (possibly small) superset V of all sets of witnesses $V_\mathbf{r}$ for $\mathbf{r} \in H$.
- Compute the set of equivalence classes for the equivalence relation: $\mathbf{r} \sim \mathbf{s}$ iff for all $\mathbf{x} \in V$ $\tau_\mathbf{r}(\mathbf{x}) = \tau_\mathbf{s}(\mathbf{x})$ and $\tau_\mathbf{r}^\star(\mathbf{x}) = \tau_\mathbf{s}^\star(\mathbf{x})$.
- Handle classes is sophisticated order to minimize computation time.

Properties:

- Much faster than previous methods by Brunotte.

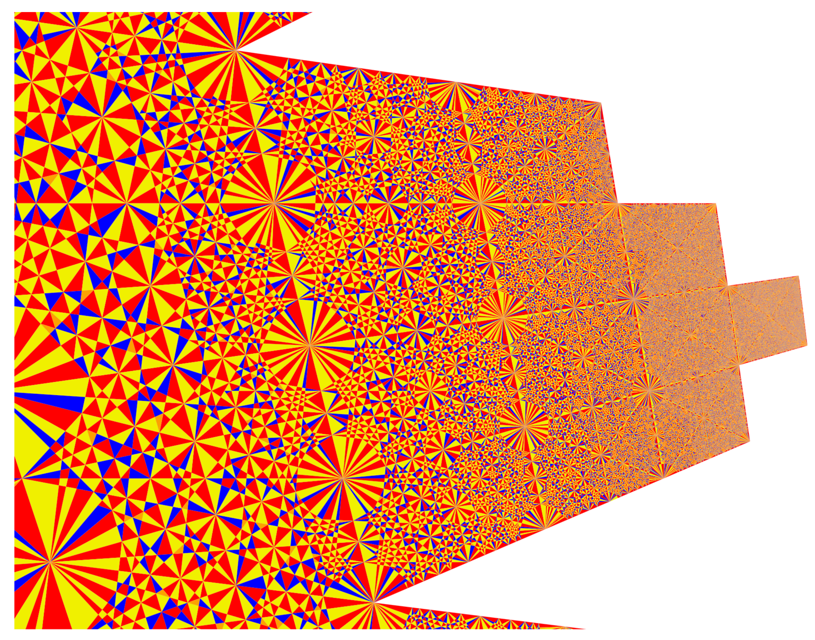

Figure. Hidden structure of $\mathcal{G}_1^{(0)}$ revealed by the equivalence classes of $\sim$.

Standard Methods

For $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ let $R_\mathbf{r}$ denote the companion matrix of $\chi_\mathbf{r}(X) = X^d + r_dX^{d-1} + \cdots + r_2X + r_1$. Then $\tau_\mathbf{r}(\mathbf{x}) = R_\mathbf{r}\mathbf{x} + \mathbf{v}_\mathbf{x}$ where $\mathbf{v}_\mathbf{x} = (0, \dots, 0, \{\mathbf{rx}\})$. Let $\rho(M)$ denote the spectral radius of a matrix.

Then
- $\mathcal{D}_d \;\subseteq \{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_\mathbf{r}) \le 1\}$
- $\mathcal{E}_d \;:= \{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_\mathbf{r}) < 1\} \subseteq \mathcal{D}_d$
- $\partial\mathcal{D}_d = \{\mathbf{r} \in \mathbb{R}^d \mid \rho(R_\mathbf{r}) = 1\}$

For a tuple π of vectors in $\mathbb{Z}^d$ let $\mathcal{P}(\pi)$ denote the set of all $\mathbf{r} \in \mathbb{R}^d$ for which π is a period of $\tau_\mathbf{r}$, i.e. $\pi = (\mathbf{x_1}, \dots, \mathbf{x_n})$, $\tau_\mathbf{r}(\mathbf{x_1}) = \mathbf{x_2}, \dots, \tau_\mathbf{r}(\mathbf{x_n}) = \mathbf{x_1}$. Then $\mathcal{P}(\pi)$ is a (possibly degenerate) convex polyhedron characterized by a finite set of linear inequalities. If $\mathcal{P}(\pi)$ is non-empty it is called a cutout polyhedron.

Theorem. $\mathcal{D}_d^{(0)} = \mathcal{D}_d \setminus \bigcup_{\pi \ne 0} \mathcal{P}(\pi)$ (Brunotte 2001)

A set $V \subseteq \mathbb{Z}^d$ is called a set of witnesses for $\mathbf{r} \in \mathbb{R}^d$ iff it is stable under $\tau_\mathbf{r}$ and $\tau_\mathbf{r}^\star := -\tau_\mathbf{r} \circ (-\mathrm{id}_{\mathbb{Z}^d})$ and contains a generating set of the group $(\mathbb{Z}^d, +)$ which is closed under taking inverses. Every such set of witnesses has the decisive property:
$\mathbf{r} \in \mathcal{D}_d^{(0)} \Leftrightarrow \forall\, \mathbf{a} \in V : \exists\, n \in \mathbb{N} : \tau_\mathbf{r}^n(\mathbf{a}) = \mathbf{0}$

Theorem. For $\mathbf{r} \in \mathrm{int}(\mathcal{D}_d)$ let

$$V_0 := \{(\pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1)\},$$
$$V_n := V_{n-1} \cup \tau_\mathbf{r}(V_{n-1}) \cup \tau_\mathbf{r}^\star(V_{n-1}) \text{ for } n \in \mathbb{N},$$
$$V_\mathbf{r} := \textstyle\bigcup_{n \in \mathbb{N}_0} V_n.$$

Then $V_\mathbf{r}$ is a finite set of witnesses for $\mathbf{r}$. (Brunotte 2001)

Results

Theorem.
(i) $\mathcal{D}_2^{(0)}$ has at least 22 connected components
(ii) The largest connected component of $\mathcal{D}_2^{(0)}$ has at least 3 holes.

(W. 2015 [4])

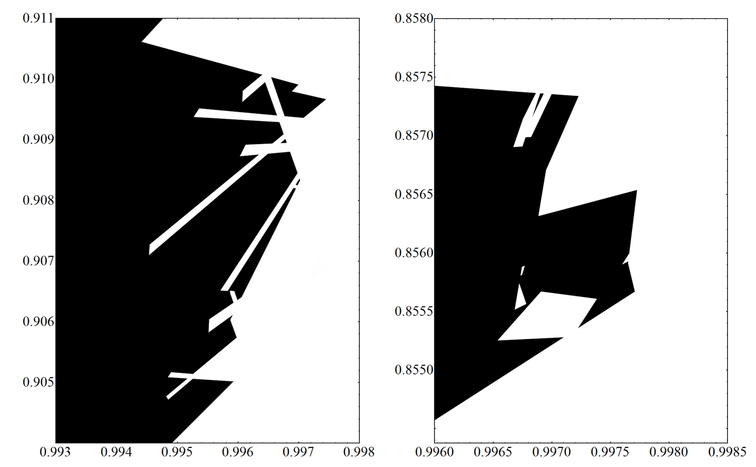

Figure. Four connected components and two holes of $\mathcal{D}_2^{(0)}$.

Conjecture. $\mathcal{G}_1^{(0)} = \mathcal{G}_C$ where $\mathcal{G}_C$ is a (neither open nor closed) polygon given by ten infinite sequences of points in $\mathbb{C}$ (and their complex conjugates). (W.)

Theorem.
(i) $\mathcal{G}_1^{(0)} \subseteq \mathcal{G}_C$
(ii) $\mathcal{G}_C \cap \{\mathbf{r} \in \mathbb{C} \mid |\mathbf{r}| \le \frac{1023}{1024}\} \subseteq \mathcal{G}_1^{(0)}$. (W. in preparation [3])

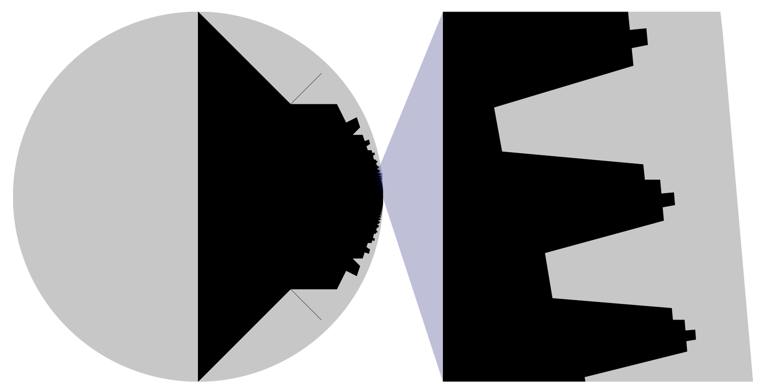

Figure. $\mathcal{G}_1^{(0)}$ in $\mathcal{G}_1$

Joint work (A. Pethő, P. Varga) on SRS over imaginary quadratic Euclidean domains in preparation.

Results

Theorem. The quotient $v_d^{(1)}/v_d^{(0)}$ is an integer for all $d \ge 2$. Furthermore we have

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \frac{P_d(3) - 2d - 1}{4} \quad \text{where}$$
$$P_d(x) := \sum_{k=0}^{d} \binom{d+k}{2k}\binom{2k}{k}\left(\frac{x-1}{2}\right)^k$$

are the Legendre polynomials. (Kirschenhofer–W. submitted [1])

Contractive Polynomials with given Signature

Definition. Let $\mathcal{E}_d$ denote the set of all coefficient vectors $(a_1, \dots, a_d) \in \mathbb{R}^d$ of contractive polynomials $x^d + a_1x^{d-1} + \cdots + a_d \in \mathbb{R}[x]$ (i.e. all roots have absolute value less than 1), and let $\mathcal{E}_d^{(s)}$ denote the subset of the coefficient vectors of those polynomials that have exactly s pairs of complex conjugate roots. Let furthermore $v_d^{(s)} := \lambda_d(\mathcal{E}_d^{(s)})$ denote the d-dimensional Lebesgue measure of $\mathcal{E}_d^{(s)}$.

Conjecture. $v_d^{(s)}/v_d^{(0)} \in \mathbb{Z}$ for all $s \ge 0$ and $d \ge 2s$.

(Akiyama–Pethő to appear)

List of Publications

[1] P. Kirschenhofer and M. Weitzer, *A number theoretic problem on the distribution of polynomials with bounded roots*, submitted.

[2] A. Pethő, P. Varga, and M. Weitzer, *On shift radix systems over imaginary quadratic Euclidean domains*, in preparation.

[3] M. Weitzer, *On the characterization of Petho's loudspeaker*, submitted.

[4] M. Weitzer, *Characterization algorithms for shift radix systems with finiteness property*, Int. J. Number Theory (2015), to appear.