Abstract

The subject of this talk is the famous Collatz conjecture which is also known as the $3n+1$ problem. Let n be any non-negative integer and apply the following transformation: If n is even divide it by 2 and if n is odd multiply it by 3 and add 1. The Collatz conjecture is the unproven observation that no matter what number n one starts with, repeated application of this transformation always eventually yields 1. Though extremely easy to formulate this problem is open since 1937. In this talk we will give an introduction to the problem, present basic facts and interpretations, and deduce and discuss first nontrivial results.

The Collatz conjecture aka the $3n + 1$ problem

An introduction to the conjecture everyone can understand
but no one can prove!

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Example: $n = 17$

$17 \mapsto 52$

The conjecture

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$17 \mapsto 52 \mapsto 26$

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$17 \mapsto 52 \mapsto 26 \mapsto 13$

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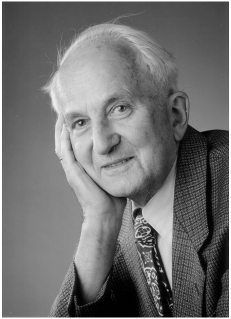
Shortcut: Combine odd step and even step to new odd step $(3n+1)/2$

Who discovered the problem?

The conjecture was first propose by Lothar Collatz (1910-1990) in 1937.



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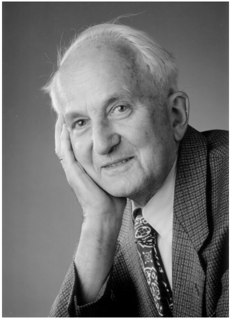

Other common names:
$3n + 1$ problem, Ulam conjecture, Kakutani's problem,
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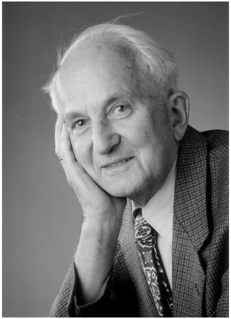

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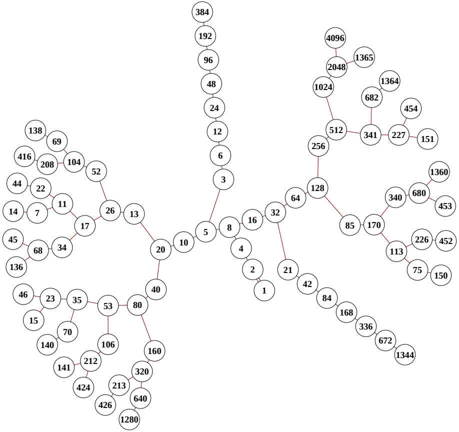
Mathematics may not be ready for such problems - Paul Erdős

The Collatz graph: Does it cover all natural numbers?



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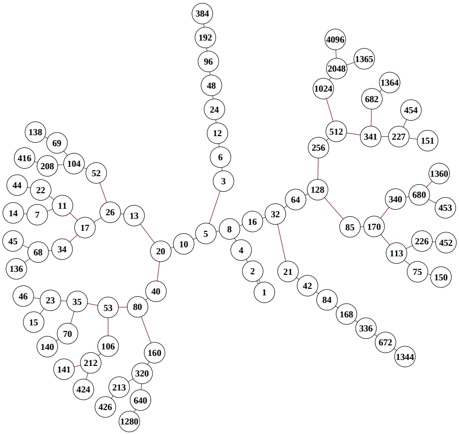

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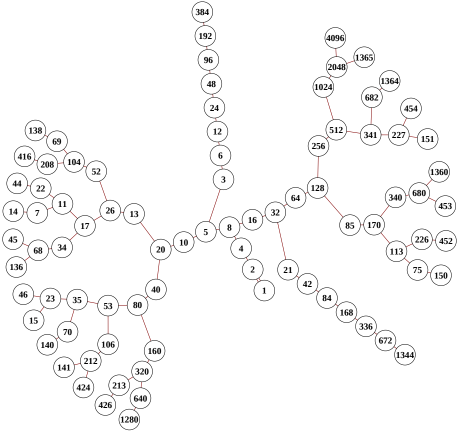

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Note: 4-digit numbers already present, but not 9!

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$$\begin{array}{r} 100110_2 + \\ 10011_2 \\ \hline 111001_2 \end{array} = 57$$

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Does repeated application of these three operations always lead to 1_2?

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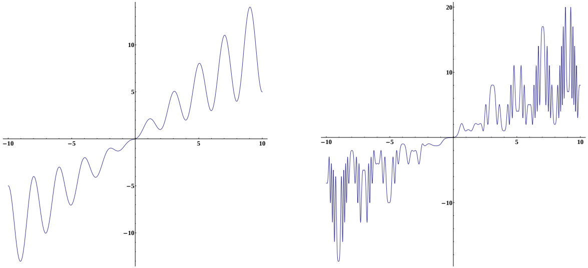
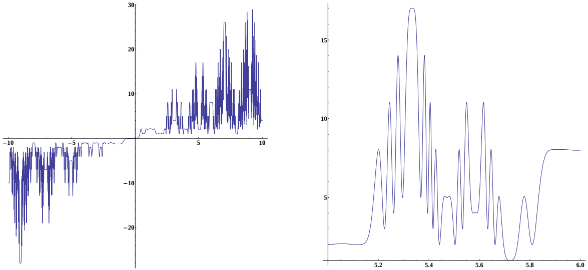
Is there a $k \in \mathbb{N}$ for every $n \in \mathbb{N}$ such that $f^k(n) = 1$?

Interpretations

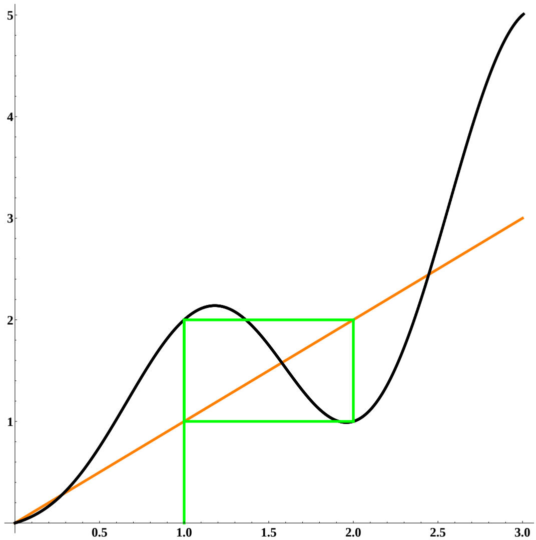


f and f^2

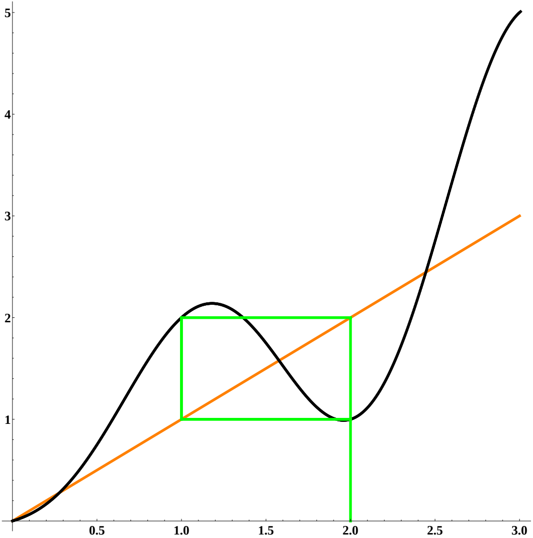


f^3

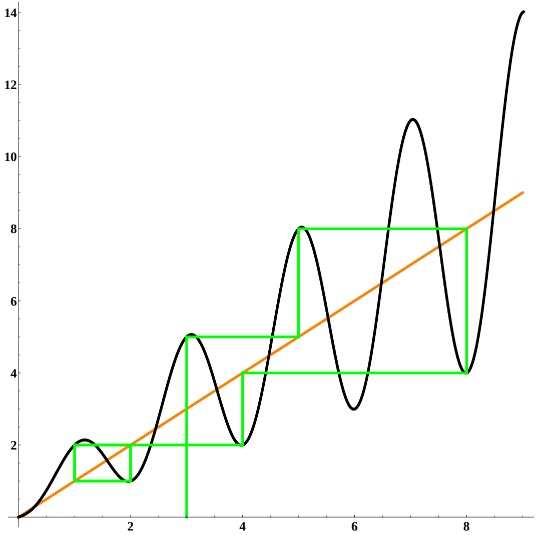
Orbit of $n = 1$:



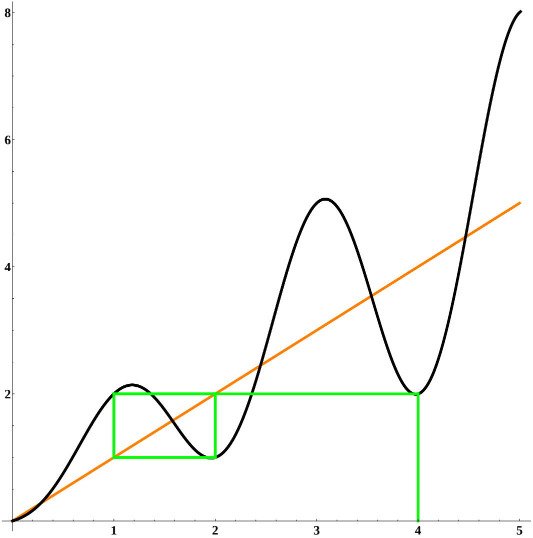
Orbit of $n = 2$:



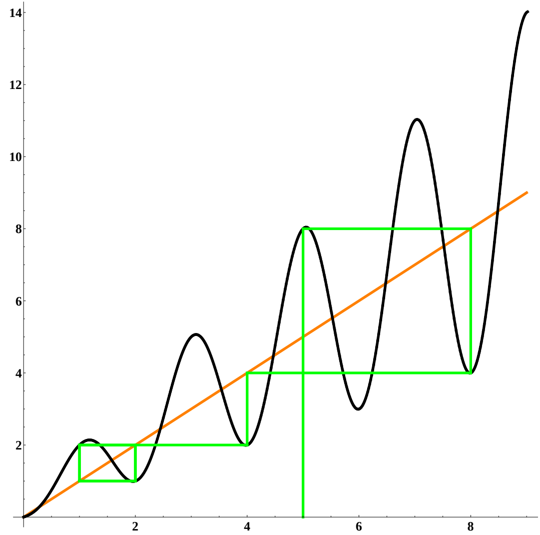
Orbit of $n = 3$:



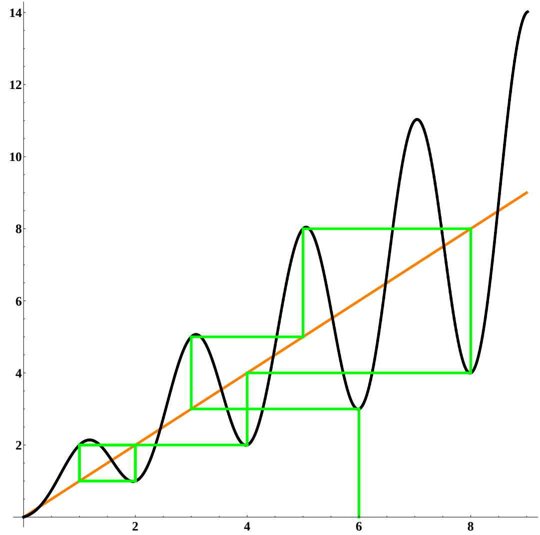
Orbit of $n = 4$:



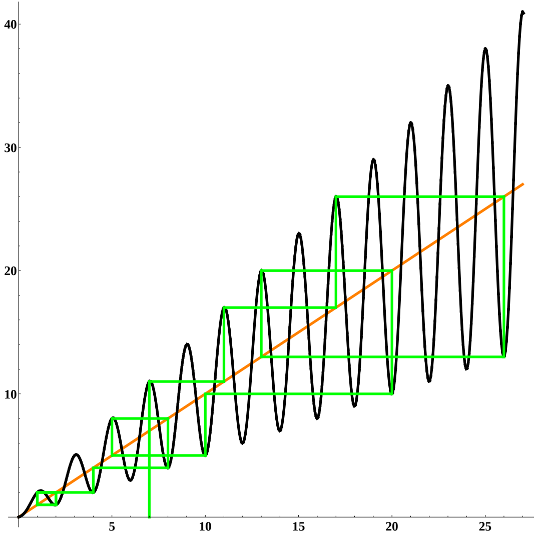
Orbit of $n = 5$:



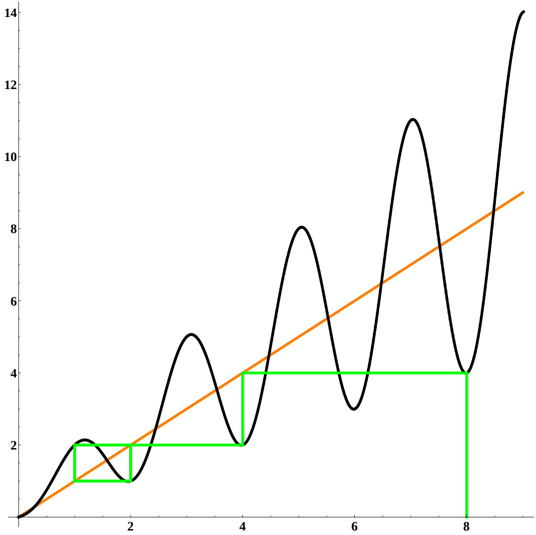
Orbit of $n = 6$:



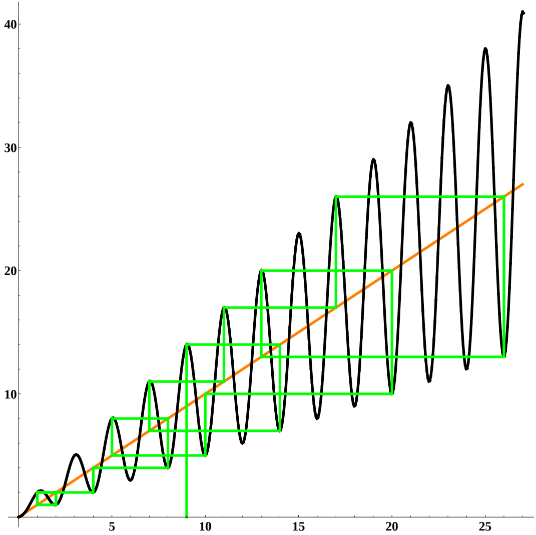
Orbit of $n = 7$:



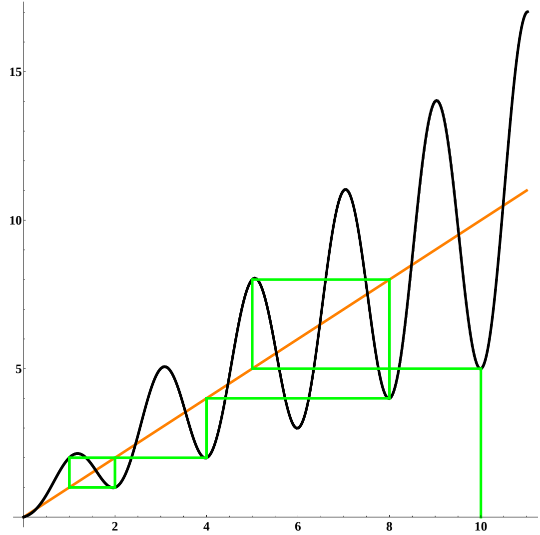
Orbit of $n = 8$:



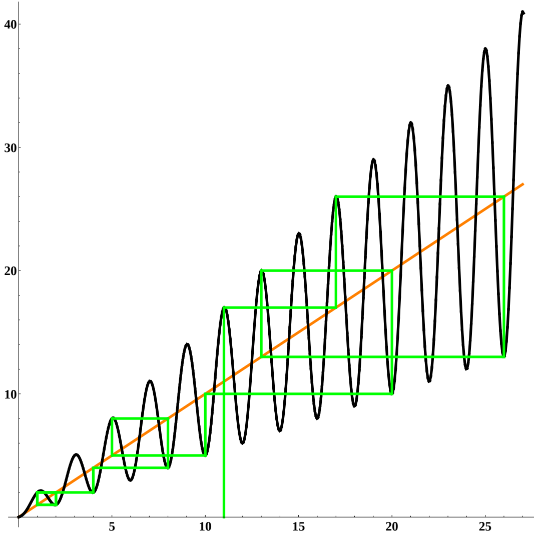
Orbit of $n = 9$:



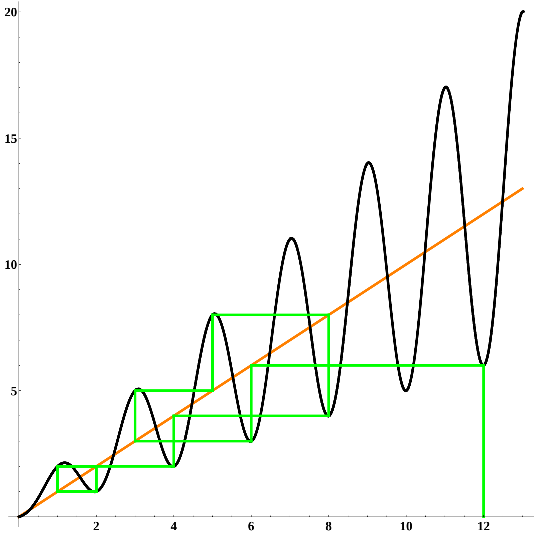
Orbit of $n = 10$:



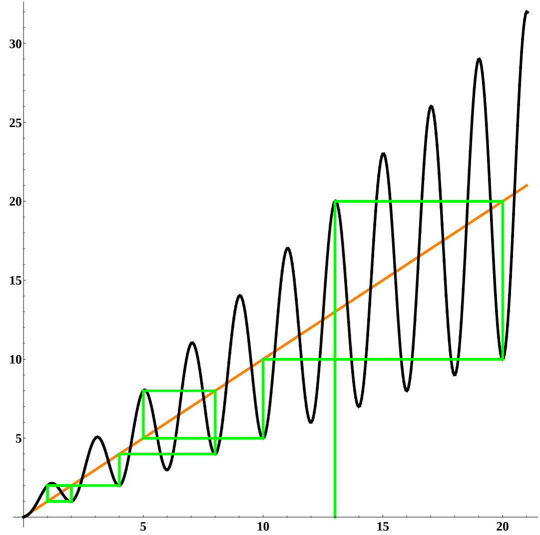
Orbit of $n = 11$:



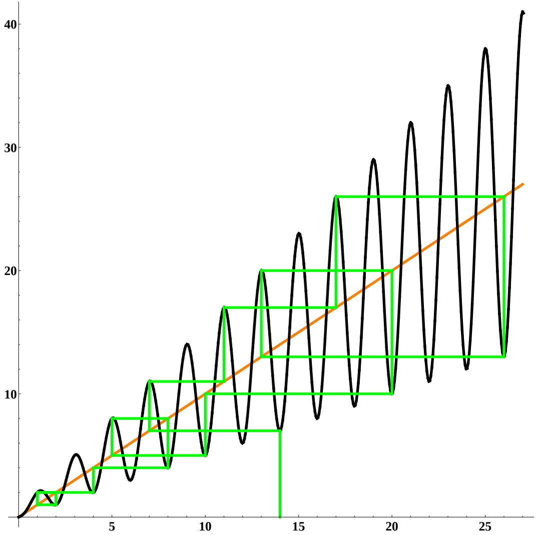
Orbit of $n = 12$:



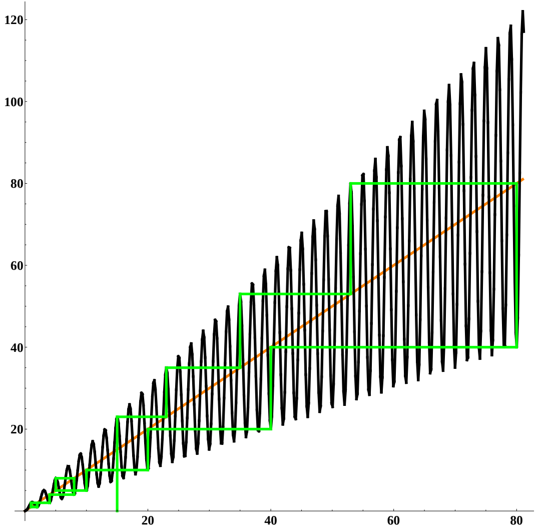
Orbit of $n = 13$:



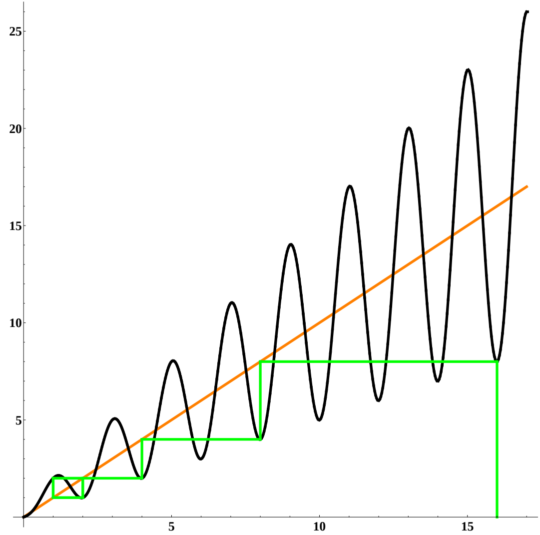
Orbit of $n = 14$:



Orbit of $n = 15$:



Orbit of $n = 16$:



f is a holomorphic function

For which $z \in \mathbb{C}$ is $\left\{ f^k(z) \mid k \in \mathbb{N} \right\}$ bounded?

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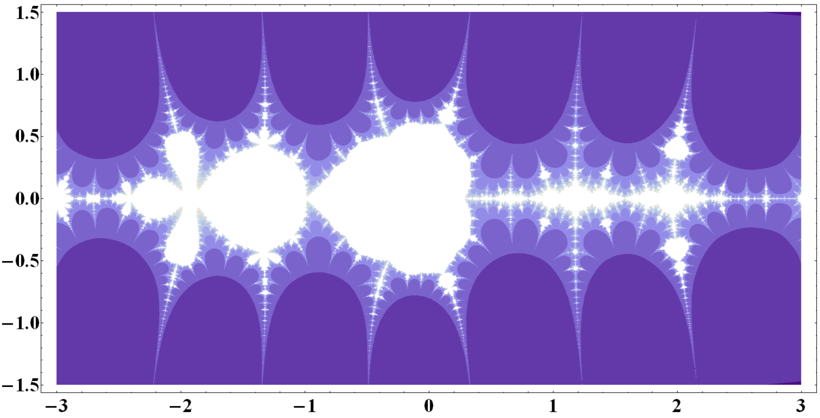

Question of divisibility of certain double-base numbers

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Later!

First observation

Question: How many steps to get to 1?

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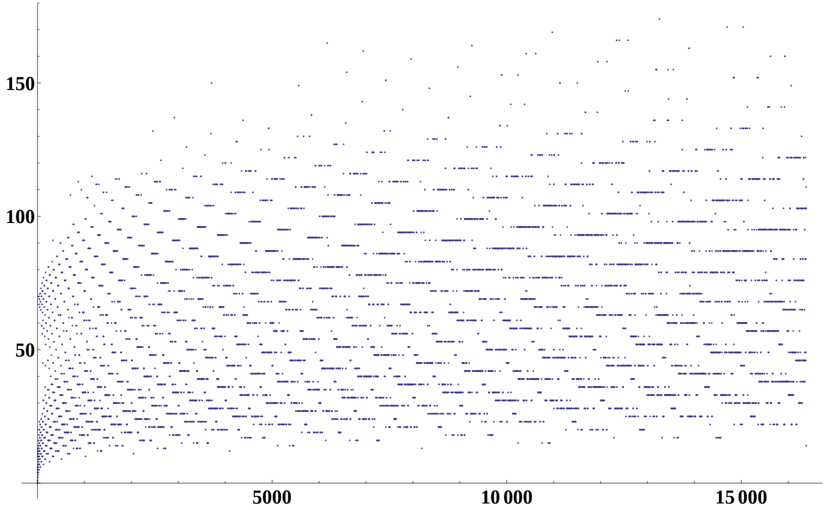

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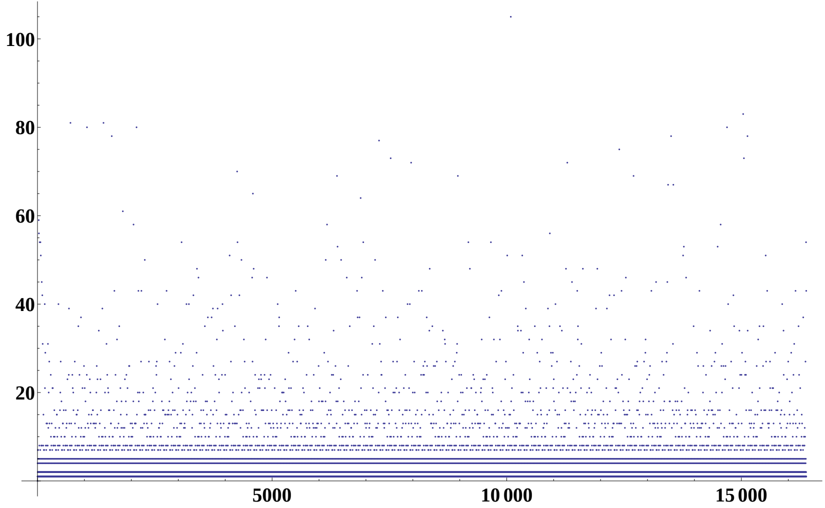

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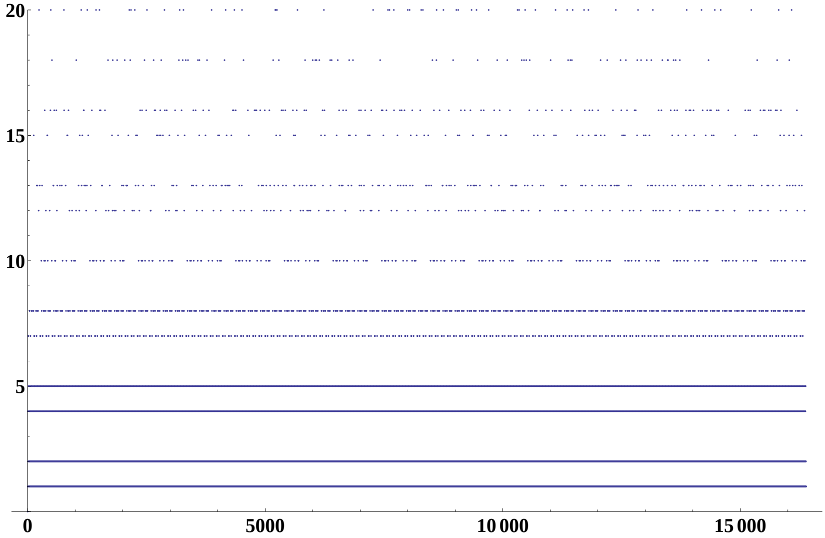

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⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰
1	1	2	1	2	1	2	1	2	1	2	⋯
2	2	1	2	1	2	1	2	1	2	1	⋯
3	3	5	8	4	2	1	2	1	2	1	⋯
4	4	2	1	2	1	2	1	2	1	2	⋯
5	5	8	4	2	1	2	1	2	1	2	⋯
6	6	3	5	8	4	2	1	2	1	2	⋯
7	7	11	17	26	13	20	10	5	8	4	⋯
8	8	4	2	1	2	1	2	1	2	1	⋯
9	9	14	7	11	17	26	13	20	10	5	⋯
10	10	5	8	4	2	1	2	1	2	1	⋯
11	11	17	26	13	20	10	5	8	4	2	⋯
12	12	6	3	5	8	4	2	1	2	1	⋯
13	13	20	10	5	8	4	2	1	2	1	⋯
14	14	7	11	17	26	13	20	10	5	8	⋯
15	15	23	35	53	80	40	20	10	5	8	⋯
16	16	8	4	2	1	2	1	2	1	2	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	0	1	2	3	4	5	6	7	8	9	⋯

Better question: How many steps to get below the starting value?

⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰
1	1	2	1	2	1	2	1	2	1	2	⋯
2	2	1	2	1	2	1	2	1	2	1	⋯
3	3	5	8	4	2	1	2	1	2	1	⋯
4	4	2	1	2	1	2	1	2	1	2	⋯
5	5	8	4	2	1	2	1	2	1	2	⋯
6	6	3	5	8	4	2	1	2	1	2	⋯
7	7	11	17	26	13	20	10	5	8	4	⋯
8	8	4	2	1	2	1	2	1	2	1	⋯
9	9	14	7	11	17	26	13	20	10	5	⋯
10	10	5	8	4	2	1	2	1	2	1	⋯
11	11	17	26	13	20	10	5	8	4	2	⋯
12	12	6	3	5	8	4	2	1	2	1	⋯
13	13	20	10	5	8	4	2	1	2	1	⋯
14	14	7	11	17	26	13	20	10	5	8	⋯
15	15	23	35	53	80	40	20	10	5	8	⋯
16	16	8	4	2	1	2	1	2	1	2	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
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For an odd integer c let

$$\begin{aligned} \mathrm{F}_c &: \mathbb{Z} \to \mathbb{Z} \\ n &\mapsto \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \mod 2 \\ \frac{3n+c}{2} & \text{if } n \equiv 1 \mod 2 \end{cases} \end{aligned}$$

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Collatz conjecture: $\forall\, n \in \mathbb{N} : \exists\, k \in \mathbb{N} : \mathrm{F}_1^k(n) = 1$

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Collatz conjecture: $\forall\, n \in \mathbb{N} : \exists\, k \in \mathbb{N} : \mathrm{F}_1^k(n) = 1$

From now on: c an arbitrary odd integer

May drop c if $c = 1$

For any integer n let

$$Z_n^{(c)} := \left(z_{n,k}^{(c)}\right)_{k\in\mathbb{N}_0} := \left(\mathrm{F}_c^k(n)\right)_{k\in\mathbb{N}_0}$$

the Collatz sequence of n (for c) and

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the Collatz signature of n (for c)

Example: $Z_{17}^{(1)} = (17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, \ldots)$

$S_{17}^{(1)} = (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, \ldots)$

$z_{17,3}^{(1)} = 20$

$s_{17,3}^{(1)} = 0$

Better question: How many steps to get below the starting value?

$Z_n^{(1)}$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	2	1	2	$\cdots$
2	2	1	2	1	$\cdots$
3	3	5	8	4	$\cdots$
4	4	2	1	2	$\cdots$
5	5	8	4	2	$\cdots$
6	6	3	5	8	$\cdots$
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8	8	4	2	1	$\cdots$
9	9	14	7	11	$\cdots$
10	10	5	8	4	$\cdots$
11	11	17	26	13	$\cdots$
12	12	6	3	5	$\cdots$
13	13	20	10	5	$\cdots$
14	14	7	11	17	$\cdots$
15	15	23	35	53	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$S_n^{(1)}$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	0	1	0	$\cdots$
2	0	1	0	1	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	0	0	$\cdots$
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7	7	11	17	26	⋯
8	8	4	2	1	⋯
9	9	14	7	11	⋯
10	10	5	8	4	⋯
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3	1	1	0	0	⋯
4	0	0	1	0	⋯
5	1	0	0	0	⋯
6	0	1	1	0	⋯
7	1	1	1	0	⋯
8	0	0	0	1	⋯
9	1	0	1	1	⋯
10	0	1	0	0	⋯
11	1	1	0	1	⋯
12	0	0	1	1	⋯
13	1	0	0	1	⋯
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⋮	⋮	⋮	⋮	⋮	⋱

Proposition

Let $n, m \in \mathbb{Z}$*, and* $k \in \mathbb{N}$*. Then*

$$\forall\, i \in \{0, \ldots, k-1\} : \left(s^{(c)}_{n,i} = s^{(c)}_{m,i} \quad \Leftrightarrow \quad n \equiv m \mod 2^k \right)$$

In words:

The first k *entries of the signatures of two integers coincide iff the numbers are congruent modulo* 2^k

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Induction on k:

$k = 1$: ✓

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Proof.

Induction on k:

$k = 1$: ✓

$k - 1 \to k$: Blackboard! (Whiteboard?) □

Question: How to compute $z_{n,k}^{(c)}$ directly if the signature of n is known?

Second observation

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Suppose $S_n^{(c)} = (0, 1, 1, 0, 1, 0, \ldots)$. What is $z_{n,6}^{(c)}$?

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Proposition

Let $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then

$$z_{n,k}^{(c)} = \frac{3^{o_{n,k}^{(c)}} \cdot n + c \cdot d_{n,k}^{(c)}}{2^k}$$

where

$o_{n,k}^{(c)}$ *is the number of odd steps among the first k steps,*

$d_{n,k}^{(c)} = \sum_{i=1}^{o_{n,k}^{(c)}} 3^{o_{n,k}^{(c)} - i} 2^{O_{n,i}^{(c)} - 1}$, *and*

$O_{n,i}^{(c)}$ *is the position of the i-th odd step*

Second result

$$z_{n,k}^{(c)} = \frac{3^{o_{n,k}^{(c)}} \cdot n + c \cdot d_{n,k}^{(c)}}{2^k}, \qquad d_{n,k}^{(c)} = \sum_{i=1}^{o_{n,k}^{(c)}} 3^{o_{n,k}^{(c)} - i} 2^{O_{n,i}^{(c)} - 1}$$

Example: $n = 17,\ k = 6$

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$o_{17,6}^{(1)} = 3$

$O_{17,1}^{(1)} = 1, \quad O_{17,2}^{(1)} = 3, \quad O_{17,3}^{(1)} = 6$

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$d_{17,6}^{(1)} = 3^2 \cdot 2^0 + 3^1 \cdot 2^2 + 3^0 \cdot 2^5 = 9 + 12 + 32 = 53$

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$$z_{17,6}^{(1)} = \frac{27 \cdot 17 + 1 \cdot 53}{64} = 8$$

For a finite signature S (i.e. $S \in \{0,1\}^k$ for a $k \in \mathbb{N}$) let $\mathrm{F}_S^{(c)} : \mathbb{R} \to \mathbb{R}$ be the function which applies even and odd steps as given in S to $x \in \mathbb{R}$

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Example: $\mathrm{F}_{(1,0,1)}^{(c)}(\pi) = \dfrac{3\dfrac{\frac{3\pi+5}{2}}{2}+5}{2}$

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$$f^{(5)}_{(1,0,0,1)} = \frac{5 \cdot (3^1 \cdot 2^0 + 3^0 \cdot 2^3)}{2^4 - 3^2} = \frac{55}{7}$$

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$$\frac{55}{7} \mapsto \frac{100}{7} \mapsto \frac{50}{7} \mapsto \frac{25}{7} \mapsto \frac{55}{7}$$

Interpretation: Question of divisibility of certain double-base numbers

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Now!

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There is a Collatz cycle with signature S iff $f_S^{(c)}$ is an integer

If $f_S^{(c)}$ is an integer then $f_S^{(c)}$ is the starting value of the cycle

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Example: $c = 1$, $S = (1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0)$

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$-17 \mapsto -25 \mapsto -37 \mapsto -55 \mapsto -82 \mapsto -41 \mapsto -61 \mapsto -91 \mapsto -136 \mapsto -68 \mapsto -34 \mapsto -17$

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Other known cycles: $0 \mapsto 0, \quad -1 \mapsto -1, \quad -5 \mapsto -7 \mapsto -10 \mapsto -5$

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Other known cycles: $0 \mapsto 0, \quad -1 \mapsto -1, \quad -5 \mapsto -7 \mapsto -10 \mapsto -5$

Is $f_S^{(1)} = \frac{d_S}{2^k - 3^{o_S}} = \frac{\sum_{i=1}^{o_S} 3^{o_S - i} 2^{O_{S,i} - 1}}{2^k - 3^{o_S}}$ a positive integer for any $S \neq (0,1), (1,0), (0,1,0,1), (1,0,1,0), \ldots$?

Third corollary: $x - \mathrm{F}_S^{(c)}(x) = \left(x - f_S^{(c)}\right) \dfrac{2^k - 3^{o_S}}{2^k}$

In particular: $\operatorname{sgn}\left(x - \mathrm{F}_S^{(c)}(x)\right) = \operatorname{sgn}\left(x - f_S^{(c)}\right) \operatorname{sgn}\left(2^k - 3^{o_S}\right)$

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Better question: How many steps to get below the starting value?

$S_n^{(1)}$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
1	1	0	1	0	$\cdots$
2	0	1	0	1	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
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7	1	1	1	0	$\cdots$
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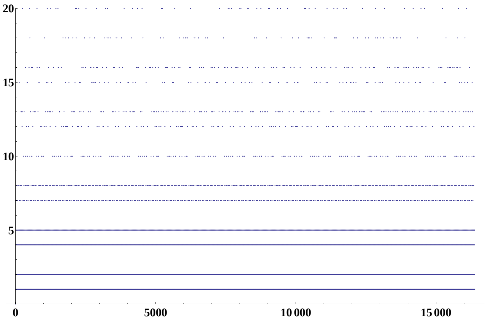
9	1	0	1	1	$\cdots$
10	0	1	0	0	$\cdots$
11	1	1	0	1	$\cdots$
12	0	0	1	1	$\cdots$
13	1	0	0	1	$\cdots$
14	0	1	1	1	$\cdots$
15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$



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$22 \mapsto 11 \mapsto 17 \mapsto 26 \mapsto 13 \mapsto 20 \mapsto 10 \mapsto 5$

Fifth consequence

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$38 \mapsto 19 \mapsto 41 \mapsto 74 \mapsto 37 \mapsto 68 \mapsto 34 \mapsto 17 \mapsto 38$

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⋮	⋮	⋮	⋮	⋮	⋰
−8	−8	−4	−2	−1	⋯
−7	−7	−10	−5	−7	⋯
−6	−6	−3	−4	−2	⋯
−5	−5	−7	−10	−5	⋯
−4	−4	−2	−1	−1	⋯
−3	−3	−4	−2	−1	⋯
−2	−2	−1	−1	−1	⋯
−1	−1	−1	−1	−1	⋯
0	0	0	0	0	⋯
1	1	2	1	2	⋯
2	2	1	2	1	⋯
3	3	5	8	4	⋯
4	4	2	1	2	⋯
5	5	8	4	2	⋯
6	6	3	5	8	⋯
7	7	11	17	26	⋯
⋮	⋮	⋮	⋮	⋮	⋱
$Z_n^{(1)}$	0	1	2	3	⋯

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−8	−8	−4	−2	−1	⋯
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0	0	0	0	0	⋯
1	1	0	0	0	⋯
2	2	1	0	0	⋯
3	3	1	0	0	⋯
4	4	2	1	0	⋯
5	5	2	1	0	⋯
6	6	3	1	0	⋯
7	7	3	1	0	⋯
⋮	⋮	⋮	⋮	⋮	⋱
$Z_n^{(\mathrm{D}_2)}$	0	1	2	3	⋯

⋮	⋮	⋮	⋮	⋮	⋰
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−6	0	1	0	0	⋯
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−4	0	0	1	1	⋯
−3	1	0	0	1	⋯
−2	0	1	1	1	⋯
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0	0	0	0	0	⋯
1	1	0	1	0	⋯
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⋮	⋮	⋮	⋮	⋮	⋱
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⋮	⋮	⋮	⋮	⋮	⋰
−8	0	0	0	1	⋯
−7	1	0	0	1	⋯
−6	0	1	0	1	⋯
−5	1	1	0	1	⋯
−4	0	0	1	1	⋯
−3	1	0	1	1	⋯
−2	0	1	1	1	⋯
−1	1	1	1	1	⋯
0	0	0	0	0	⋯
1	1	0	0	0	⋯
2	0	1	0	0	⋯
3	1	1	0	0	⋯
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5	1	0	1	0	⋯
6	0	1	1	0	⋯
7	1	1	1	0	⋯
⋮	⋮	⋮	⋮	⋮	⋱
$S_n^{(\mathrm{D}_2)}$	0	1	2	3	⋯

$S_n^{(1)}$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
−8	0	0	0	1	$\cdots$
−7	1	0	1	1	$\cdots$
−6	0	1	0	0	$\cdots$
−5	1	1	0	1	$\cdots$
−4	0	0	1	1	$\cdots$
−3	1	0	0	1	$\cdots$
−2	0	1	1	1	$\cdots$
−1	1	1	1	1	$\cdots$
0	0	0	0	0	$\cdots$
1	1	0	1	0	$\cdots$
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3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	0	0	$\cdots$
6	0	1	1	0	$\cdots$
7	1	1	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$S_n^{(\mathrm{D}_2)}$	0	1	2	3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
−8	0	0	0	1	$\cdots$
−7	1	0	0	1	$\cdots$
−6	0	1	0	1	$\cdots$
−5	1	1	0	1	$\cdots$
−4	0	0	1	1	$\cdots$
−3	1	0	1	1	$\cdots$
−2	0	1	1	1	$\cdots$
−1	1	1	1	1	$\cdots$
0	0	0	0	0	$\cdots$
1	1	0	0	0	$\cdots$
2	0	1	0	0	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	1	0	$\cdots$
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7	1	1	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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Base 2 expansion: All signatures ultimately periodic with periods (0) and (1)

Collatz: All signatures probably ultimately periodic with periods (0), (1), $(0, 1)$, $(1, 1, 0)$, and $(1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0)$

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Which parameters control what orbits one can get?

Permutation towers

Translate between two number systems F and G

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$$\psi_{\mathrm{F},k} : \mathbb{Z}/2^k\mathbb{Z} \to \{\text{Signatures of length } k\}$$
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Number system: Assume $\psi_{\mathrm{F},k}$ well-defined and bijective for all $k \in \mathbb{N}_0$, so

$$\forall\, i \in \{0, \dots, k-1\} : \left(s_{n,i}^{(\mathrm{F})} = s_{m,i}^{(\mathrm{F})} \quad \Leftrightarrow \quad n \equiv m \mod 2^k \right)$$

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Define

$$\pi_{\mathrm{F},\mathrm{G},k} := {\psi_{\mathrm{G},k}}^{-1} \circ \psi_{\mathrm{F},k}$$
$$T_{\mathrm{F},\mathrm{G}} := (\pi_{\mathrm{F},\mathrm{G},k})_{k \in \mathbb{N}_0}$$

$\pi_{\mathrm{F},\mathrm{G},k}$ is a permutation of $\mathbb{Z}/2^k\mathbb{Z}$

$T_{\mathrm{F},\mathrm{G}}$ is a "permutation tower"

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Assume: $\pi_{\mathrm{F},\mathrm{G},k} = \sigma_{k,1} \circ \ldots \circ \sigma_{k,r_k}$ for all $k \in \mathbb{N}_0$ (cycle decomposition)

$\sigma_{k,i} = (a_{k,i,1}, \ldots, a_{k,i,s_{k,i}})$ for all $k \in \mathbb{N}_0$ and $i \in \{1, \ldots, r_k\}$

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Proposition

For every $\sigma_{k,i}$ either:

There are σ_{k+1,j_1} and σ_{k+1,j_2} such that

$s_{k+1,j_1} = s_{k+1,j_2} = s_{k,i}$

$a_{k,i,1} \equiv a_{k+1,j_1,1} \mod 2^k, \ldots, a_{k,i,s_{k,i}} \equiv a_{k+1,j_1,s_{k,i}} \mod 2^k$ *(w.l.o.g.)*

$a_{k,i,1} \equiv a_{k+1,j_2,1} \mod 2^k, \ldots, a_{k,i,s_{k,i}} \equiv a_{k+1,j_2,s_{k,i}} \mod 2^k$

or:

There is a $\sigma_{k+1,j}$ such that

$s_{k+1,j} = 2s_{k,i}$

$a_{k,i,1} \equiv a_{k+1,j,1} \mod 2^k, \ldots, a_{k,i,s_{k,i}} \equiv a_{k+1,j,s_{k,i}} \mod 2^k$

$a_{k,i,1} \equiv a_{k+1,j,s_{k,i}+1} \mod 2^k, \ldots, a_{k,i,s_{k,i}} \equiv a_{k+1,j,2s_{k,i}} \mod 2^k$

Example: $\mathrm{F} = \mathrm{F}_{1,4}$ ($\mathrm{F}_{c,s}$: add s to n, then F_c) and $\mathrm{G} = \mathrm{D}_2$

Collatz the number system

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$S_n^{(\mathrm{F}_{1,4})}$	0	1	2	3	⋯
⋮	⋮	⋮	⋮	⋮	⋰
0	0	0	1	0	⋯
1	1	0	0	0	⋯
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3	1	1	1	0	⋯
4	0	0	0	1	⋯
5	1	0	1	1	⋯
6	0	1	0	0	⋯
7	1	1	0	1	⋯
8	0	0	1	1	⋯
9	1	0	0	1	⋯
10	0	1	1	1	⋯
11	1	1	1	1	⋯
12	0	0	0	0	⋯
13	1	0	1	0	⋯
14	0	1	0	1	⋯
15	1	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋱

$S_n^{(\mathrm{D}_2)}$	0	1	2	3	⋯
⋮	⋮	⋮	⋮	⋮	⋰
0	0	0	0	0	⋯
1	1	0	0	0	⋯
2	0	1	0	0	⋯
3	1	1	0	0	⋯
4	0	0	1	0	⋯
5	1	0	1	0	⋯
6	0	1	1	0	⋯
7	1	1	1	0	⋯
8	0	0	0	1	⋯
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10	0	1	0	1	⋯
11	1	1	0	1	⋯
12	0	0	1	1	⋯
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15	1	1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋱

$S_n^{(\mathrm{D}_2)}$	0	1	2	3	⋯
⋮	⋮	⋮	⋮	⋮	⋰
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15	1	1	1	1	⋯
⋮	⋮	⋮	⋮	⋮	⋱

$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,0} = (0)$

$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,1} = (0,1)$

$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,2} = (0,1,2,3)$

$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,3} = (4,1,6,7,0,5,2,3)$

$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,4} = (4,1,6,7,8,13,2,11,12,9,14,15,0,5,10,3)$

$$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,0} = (0)$$
$$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,1} = (0, 1)$$
$$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,2} = (0, 1, 2, 3)$$
$$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,3} = (4, 1, 6, 7, 0, 5, 2, 3)$$
$$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,4} = (4, 1, 6, 7, 8, 13, 2, 11, 12, 9, 14, 15, 0, 5, 10, 3)$$

$$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,0} = (0)$$
$$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,1} = (0, 1)$$
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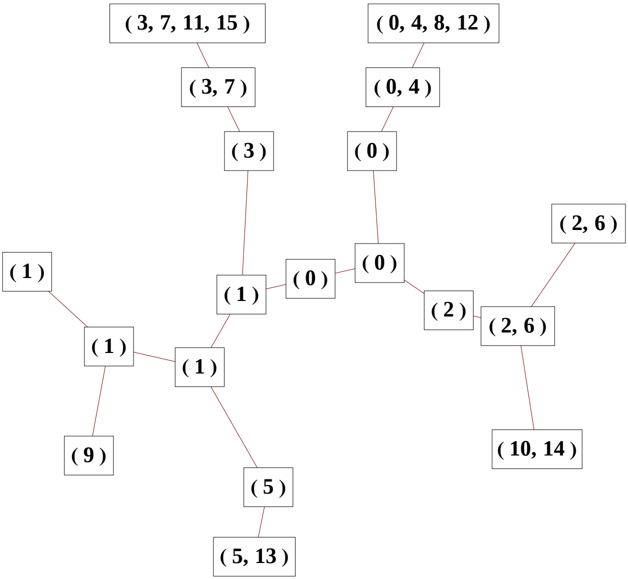
$$\pi_{\mathrm{F}_{1,4},\mathrm{D}_2,4} = (4, 1, 6, 7, 8, 13, 2, 11, 12, 9, 14, 15, 0, 5, 10, 3)$$

$k = 0$: (0)
$k = 1$: $(0), (1)$
$k = 2$: $(0), (2), (1), (3)$
$k = 3$: $(0, 4), (2, 6), (1), (5), (3, 7)$
$k = 4$: $(0, 4, 8, 12), (2, 6), (10, 14), (1), (9), (5, 13), (3, 7, 11, 15)$


(3, 7, 11, 15)
(0, 4, 8, 12)
(3, 7)
(0, 4)
(3)
(0)
(2, 6)
(1)
(0)
(0)
(1)
(2)
(2, 6)
(1)
(1)
(9)
(10, 14)
(5)
(5, 13)

Graphs for different values of c (row) and s (column)