Volumes of Projections

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(1) Introduction

If a linear transformation $A \in \mathbb{R}^{d\times d}$ is applied to a polytope $\mathcal{P}$ in $\mathbb{R}^d$, the volume of $\mathcal{P}$ changes by a factor of $|\det(A)|$:

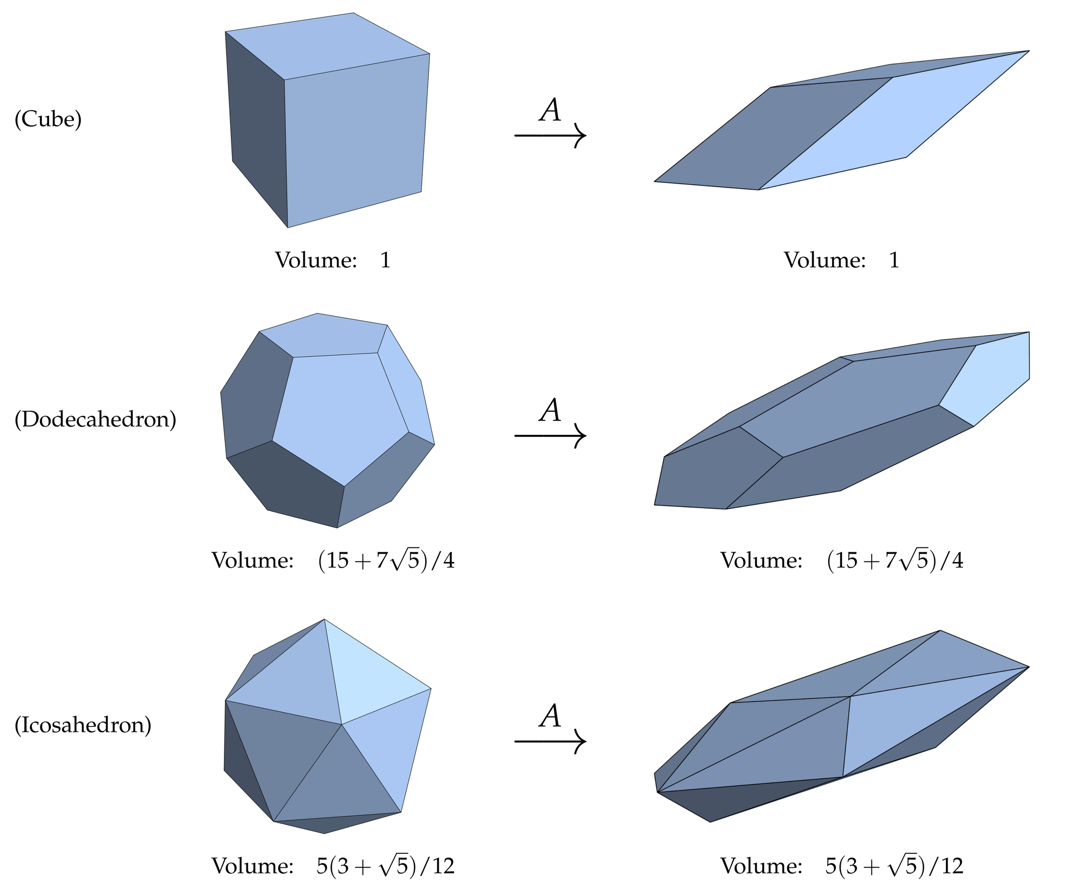

Figure. No change of volume for $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2\cos(\pi/8) & 1/2\sin(\pi/8) \\ 0 & -2\sin(\pi/8) & -1/2\cos(\pi/8)\end{pmatrix}$ as $\det(A) = 1$.

The change of volume only depends on A but not on $\mathcal{P}$.

If $A \in \mathbb{R}^{d\times D}$ where $d < D$, this must not be the case:

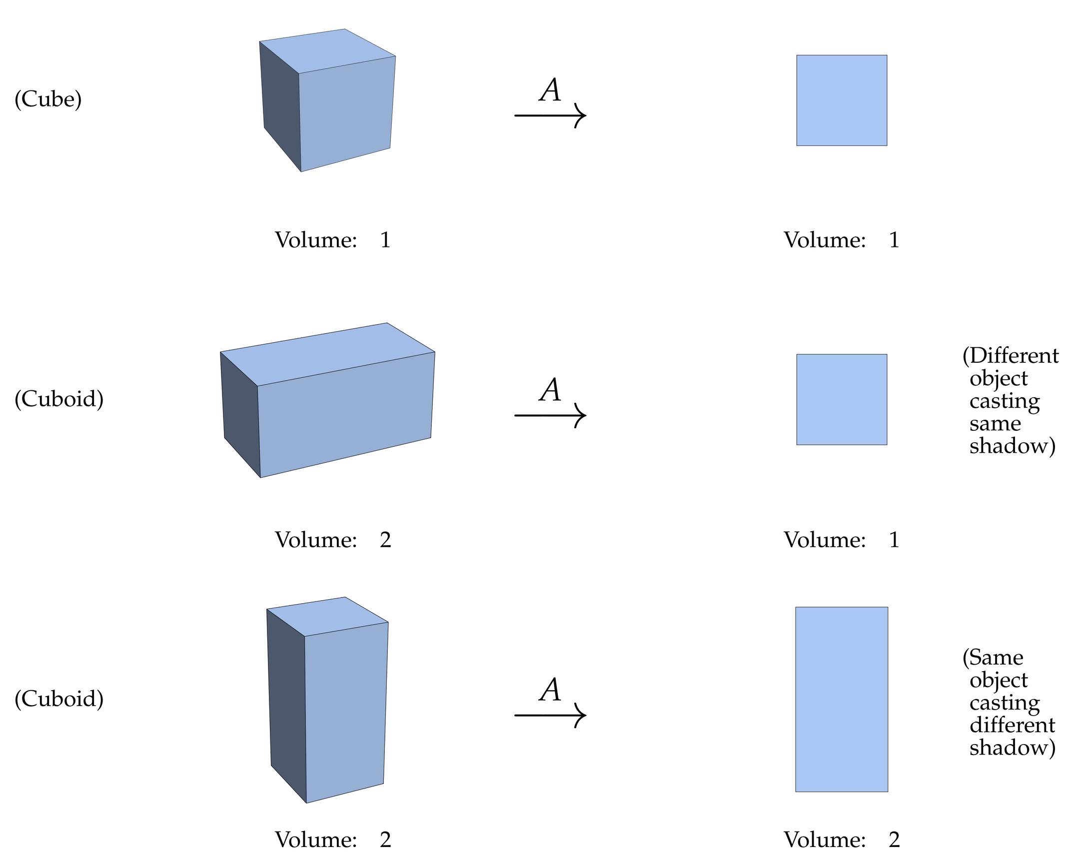

Figure. Changing volumes for $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix}$.

The "volume" (area) of the shadow that an object casts, is dependent on the object and its orientation.

(2) Triangulations

In general, the computation of the volume of a polytope $\mathcal{P}$ in $\mathbb{R}^d$ is a hard problem as it requires the computation of a triangulation of $\mathcal{P}$. A triangulation is a decomposition of $\mathcal{P}$ into simplices in $\mathbb{R}^d$ (non-degenerate polytopes that have exactly $d+1$ vertices) such that the intersection of any two simplices is either empty or a common face of both.

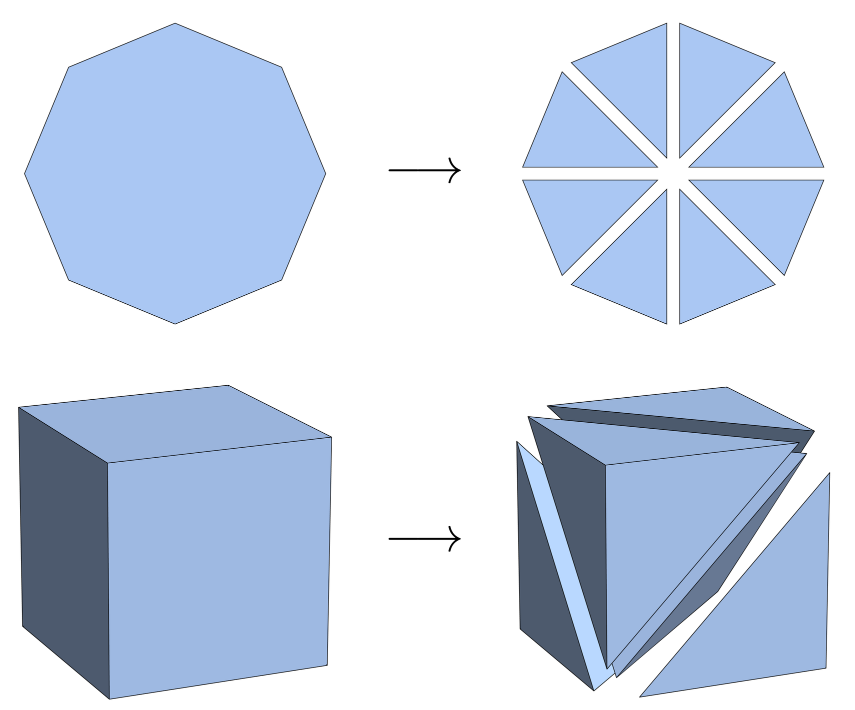

Figure. Triangulations of a regular octagon and a cube.

The volume of $\mathcal{P}$ is then given by the sum of the volumes of the simplices which can easily be computed.

(3) Main Goal and Motivation

Our main objective is to generalize the relation $\operatorname{vol}(A\mathcal{P}) = |\det(A)|\operatorname{vol}(\mathcal{P})$ for $A \in \mathbb{R}^{d\times d}$ (as explained in the introduction) to the situation where $A \in \mathbb{R}^{d\times D}$ with $d < D$. Any such matrix A can be written as a product $A = P_{D,d}B$ of the projection matrix

$$P_{D,d} := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{d\times D}$$

and some matrix $B \in \mathbb{R}^{D\times D}$. We can thus assume without loss of generality that $A = P_{D,d}$. In this situation it makes sense to speak of the polytope $\mathcal{P}$ and its "shadow" $\mathcal{S} := A\mathcal{P}$ and to ask what is the relation between the volumes of $\mathcal{P}$ and $\mathcal{S}$.

While the above question is of interest on its own, our original motivation to study the relation between the volumes of polytopes and their shadows was a specific number-theoretic one. Several problems in number theory are related to the volumes of the members of infinite families of polytopes, with examples being the family of Birkhoff polytopes or the family of Permutahedra. Another family of polytopes, the volumes of which is of number-theoretic interest, was introduced by C. Fuchs, R. Tichy, and V. Ziegler in 2009. The volumes of these polytopes are related to an arithmetic constant that was studied by G. R. Everest in 1990. While the direct computation of these volumes turned out to be very difficult we were able to recognize the complicated polytopes of the family as the shadows of comparatively simple polytopes from another family, the volumes of which can be computed very easily.

(4) Main Result

Lifting Theorem.

Let

- $\mathcal{P}$ be a polytope in $\mathbb{R}^D$
- $\mathcal{S} := P_{D,d}\mathcal{P}$ its "shadow" in $\mathbb{R}^d$ such that
  - $\mathbf{0}$ is an interior point of $\mathcal{S}$
  - exactly $D-d+1$ of the vertices of $\mathcal{P}$ are projected to $\mathbf{0}$
  - the other vertices of $\mathcal{P}$ are in one-to-one correspondence with the vertices of $\mathcal{S}$
  - $\mathcal{P}$ is contained in the union of all simplices in $\mathbb{R}^D$ that are spanned by any d vertices of $\mathcal{P}$ and the $D-d+1$ vertices which are projected to $\mathbf{0}$
- $\mathfrak{S}$ a triangulation of $\mathcal{S}$ where all simplices in $\mathfrak{S}$ are spanned by any d vertices of $\mathcal{S}$ and $\mathbf{0}$
- $\mathfrak{T}$ lift of $\mathfrak{S}$ to $\mathbb{R}^D$ which is gained by replacing all vertices of a simplex in $\mathfrak{S}$ that are distinct from $\mathbf{0}$ by their corresponding vertices of $\mathcal{P}$ and adding the $D-d+1$ vertices which are projected to $\mathbf{0}$.

Then

- $\mathfrak{T}$ is a triangulation of $\mathcal{P}$
- there is a computable constant $c \in \mathbb{R}$ such that for any simplex σ in $\mathfrak{S}$ and the corresponding lifted simplex τ in $\mathfrak{T}$ one gets $\operatorname{vol}(\tau) = c\operatorname{vol}(\sigma)$
- $\operatorname{vol}(\mathcal{P}) = c\operatorname{vol}(\mathcal{S})$.

(Kerber, Tichy, Weitzer)

(5) Examples

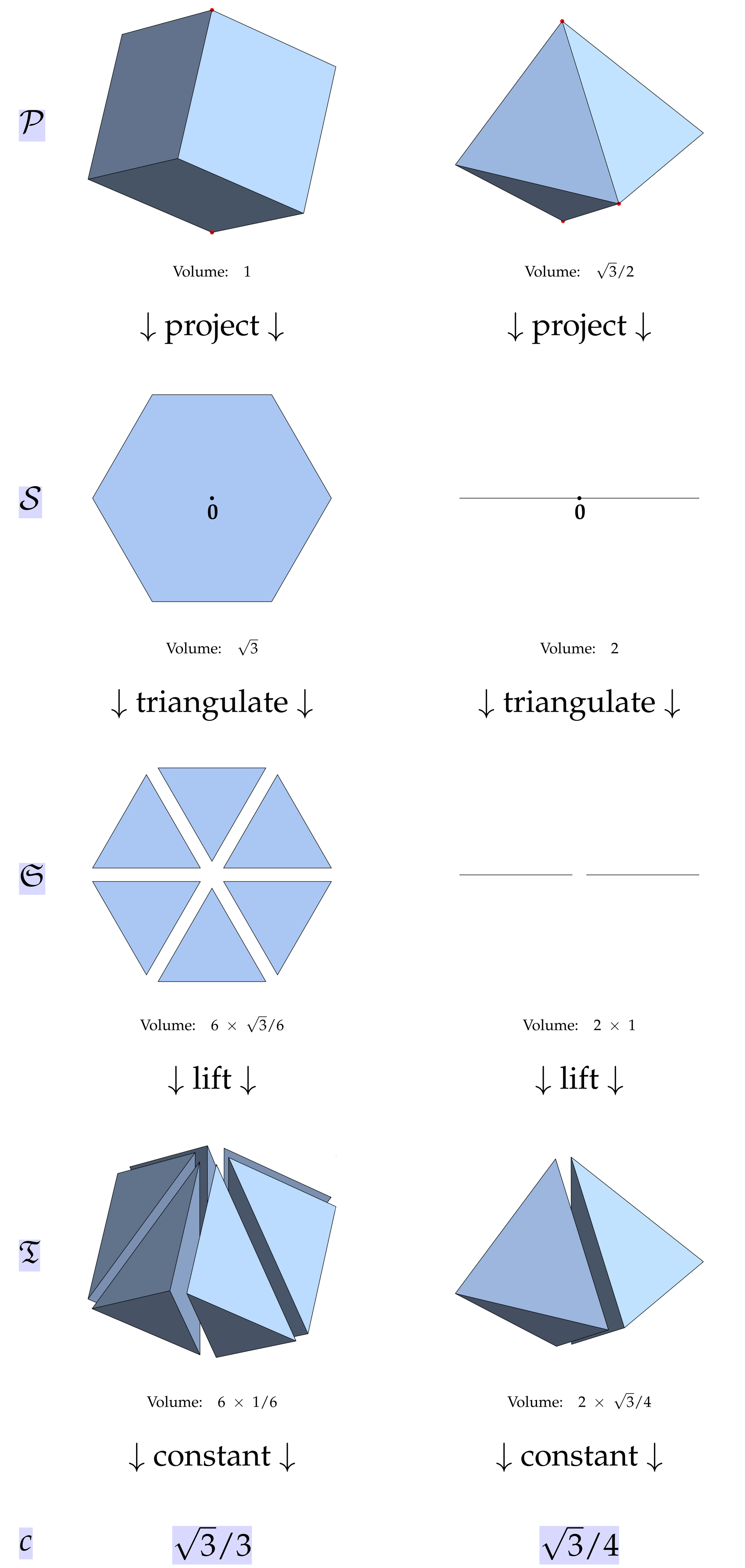

Figure. Two examples satisfying the conditions of the Lifting Theorem. Vertices of $\mathcal{P}$ that are projected to $\mathbf{0}$ by $P_{D,d}$ are marked in red. Left column: $D=3, d=2$. Right column: $D=3, d=1$.

The above examples demonstrate the statement of the Lifting Theorem. The special triangulation $\mathfrak{S}$ of the "shadow" $\mathcal{S}$, the simplices of which have $\mathbf{0}$ as a common vertex, can be lifted to a triangulation $\mathfrak{T}$ of the "shadow-casting" polytope $\mathcal{P}$, which satisfies that its simplices have all the vertices of $\mathcal{P}$, that are projected to $\mathbf{0}$, in common. It is easy to show that such a triangulation of $\mathcal{S}$ always exists if $\mathbf{0}$ is an interior point of $\mathcal{S}$. Furthermore it can be seen that the volumes of corresponding simplices in $\mathfrak{S}$ and $\mathfrak{T}$ differ by a constant factor c. It follows that the volumes of $\mathcal{S}$ and $\mathcal{P}$ differ by the same factor. c only depends on the vertices of $\mathcal{P}$ that are projected to $\mathbf{0}$, and can be computed very easily. Indeed, just as in the case where $A \in \mathbb{R}^{d\times d}$ as discussed in the introduction, it is a simple determinant of a certain matrix defined by these vertices. Thus, if the conditions of the Lifting Theorem are met and the volume of either $\mathcal{S}$ or $\mathcal{P}$ is known, the respective other volume can be computed by simply multiplying or dividing by c. Considering the above examples, this means that the volume of $\mathcal{S}$ only depends on the volume of $\mathcal{P}$ (or vice versa) and on the vertices marked in red. While the above examples only involve fairly simple bodies and shapes, the volumes of which can be computed directly, there are other examples (like the one indicated in (3)) where the computation of the volume of either $\mathcal{S}$ or $\mathcal{P}$ is significantly easier than the computation of the respective other volume.