1.1) (a) $(\lambda x.(((xz)y)(xx))) = \lambda x.xzy(xx)$

?

(b) $((\lambda x.(\lambda y.(\lambda z.(z((xy)z)))))(\lambda u.u)) = \lambda xyz.z(xyz)(\lambda u.u)$

1.2) $\lambda x.x(\lambda x.x) =_\alpha ?$ (a) $\lambda y.y(\lambda x.x)$ ✓

?

(b) $\lambda y.y(\lambda x.y)$ ×

(c) $\lambda y.y(\lambda y.x)$ ×

1.3) $\lambda x.x(\lambda z.y) =_\alpha \lambda z.z(\lambda z.y)$

(Proof: $\underset{D1.5.2(3a)}{\lambda z.y} =_\alpha \underset{D1.5.2(2)}{\lambda z.y} \Rightarrow$ )

Proof: $\lambda x.x(\lambda z.y) =_\alpha \lambda x.x(\lambda u.y) =_\alpha \lambda z.z(\lambda u.y) =_\alpha \lambda z.z(\lambda z.y)$ ✓

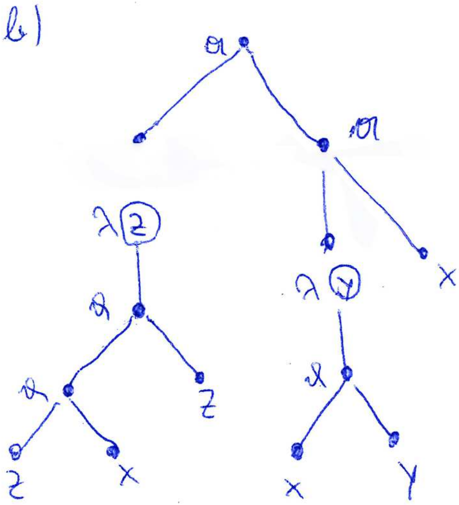
1.4) $U := (\lambda z.zxz)((\lambda y.xy)x)$

(a) $Sub(U) = \{(\lambda z.zxz)((\lambda y.xy)x), \lambda z.zxz, (\lambda y.xy)x, zxz, \lambda y.xy, x, zx, z, xy, z, x, x, y\}$

multiset

$Sub(U) = \{(\lambda z.zxz)((\lambda y.xy)x), \lambda z.zxz, zxz, zx, z, x, z, (\lambda y.xy)x, \lambda y.xy, xy, x, y, x\}$

(b)




(c) $FV(U) = FV(\lambda z.zxz) \cup FV((\lambda y.xy)x) = (FV(zxz)\setminus\{z\}) \cup FV(\lambda y.xy) \cup FV(x)$
$= \ldots = \{x\} \cup \{x\} \cup \{x\} = \underline{\{x\}}$

(d) $U =_\alpha$ $(\lambda y.yxy)((\lambda z.xz)x)$ ✓ Barendregt convention: ✓

$(\lambda x.xyx)((\lambda z.yz)y)$ × ✓

$(\lambda y.yxy)((\lambda y.xy)x)$ ✓ ×

$(\lambda v.(vx)v)((\lambda u.uv)x)$ × ×

1.5) (a) $(\lambda x.y(\lambda y.xy))[y:=\lambda z.zx] \equiv (\lambda u.y(\lambda y.uy))[y:=\lambda z.zx] \equiv \lambda u.(\lambda z.zx)(\lambda y.uy)$

(b) $((xyx)[x:=y])[y:=z] \equiv (yyy)[y:=z] \equiv zzz$

(c) $((\lambda x.xyz)[x:=y])[y:=z] \equiv (\lambda x.xyz)[y:=z] \equiv \lambda x.xzz$

(d) $(\lambda y.yyx)[x:=yz] \equiv (\lambda u.uux)[x:=yz] \equiv \lambda u.uuyz$

1.6) Find counterexample for $M[x:=N, y:=L] \equiv M[x:=N][y:=L]$

$x[x:=y, y:=x] \equiv y$

$x[x:=y][y:=x] \equiv y[y:=x] \equiv x$

1.7) $U := (\lambda z.zxz)((\lambda y.xy)x)$

(a) Redexes: $(\lambda z.zxz)((\lambda y.xy)x)$, $(\lambda y.xy)x$

(b) Reduction paths, β-normal form:

$(\lambda z.zxz)((\lambda y.xy)x)$

$((\lambda y.xy)x)x((\lambda y.xy)x)$ $(\lambda z.zxz)(xx)$

$(xx)x((\lambda y.xy)x)$ $((\lambda y.xy)x)x(xx)$

$(xx)x(xx)$

$=$

$xxx(xx)$

1.8) $(\lambda x.xx)y \neq_\beta (\lambda xy.yx)xx$

Proof: $(\lambda x.xx)y =_\beta yy$

$(\lambda xy.yx)xx =_\beta (\lambda y.yx)x =_\beta xx$

1.9) $K := \lambda xy.x$

$S := \lambda xyz.xz(yz)$

(a) $P, Q, R \in \Lambda$

$KPQ \twoheadrightarrow_\beta P$, $SPQR \twoheadrightarrow_\beta PR(QR)$

Proof: $(\lambda xy.x)PQ =_\beta (\lambda y.P)Q =_\beta P$ ✓

$(\lambda xyz.xz(yz))PQR =_\beta (\lambda yz.Pz(yz))QR =_\beta (\lambda z.Pz(Qz))R =_\beta PR(QR)$ ✓

(b) $I := \lambda x.x \Rightarrow SKK \twoheadrightarrow_\beta I$

$(\lambda xyz.xz(yz))(\lambda xy.x)(\lambda xy.x) =_\alpha (\lambda xyz.xz(yz))(\lambda uv.u)(\lambda rs.r)$

$\rightarrow_\beta (\lambda yz.(\lambda uv.u)z(yz))(\lambda rs.r)$

$\rightarrow_\beta \lambda z.(\lambda uv.u)z((\lambda rs.r)z)$

$\twoheadrightarrow_\beta \lambda z.(\lambda v.z)(\lambda s.z) \rightarrow_\beta \lambda z.z =_\alpha I$ ✓

(c) $B := S(KS)K \Rightarrow BUVW \twoheadrightarrow_\beta U(VW)$, $U, V, W \in \Lambda$

$(\lambda xyz.xz(yz))((\lambda xy.x)(\lambda xyz.xz(yz)))(\lambda xy.x)UVW \rightarrow_\beta$

$\rightarrow_\beta (\lambda xyz.xz(yz))(\lambda y.(\lambda xyz.xz(yz)))(\lambda xy.x)UVW \rightarrow_\beta$

$\rightarrow_\beta (\lambda yz.(\lambda y.(\lambda xyz.xz(yz)))z(yz))(\lambda xy.x)UVW \rightarrow_\beta$

$\rightarrow_\beta (\lambda yz.(\lambda xyz.xz(yz))(yz))(\lambda xy.x)UVW \rightarrow_\beta$

$\rightarrow_\beta (\lambda yz.(\lambda uv.(yz)v(uv)))(\lambda xy.x)UVW \rightarrow_\beta$

$\rightarrow_\beta (\lambda z.(\lambda uv.((\lambda xy.x)z)v(uv)))UVW \twoheadrightarrow_\beta (\lambda z.(\lambda uv.z(uv)))UVW \twoheadrightarrow_\beta U(VW)$ ✓

(d) $SSSKK =_\beta SKKK$

$(\lambda x_1y_1z_1.x_1z_1(y_1z_1))(\lambda x_2y_2z_2.x_2z_2(y_2z_2))(\lambda x_3y_3z_3.x_3z_3(y_3z_3))(\lambda x_4y_4.x_4)(\lambda x_5y_5.x_5)$

$\to_\beta (\lambda y_1z_1.(\lambda x_2y_2z_2.x_2z_2(y_2z_2))z_1(y_1z_1))(\lambda x_3y_3z_3.x_3z_3(y_3z_3))(\lambda x_4y_4.x_4)(\lambda x_5y_5.x_5)$

$\twoheadrightarrow_\beta (\lambda y_1z_1.(\lambda z_2.z_1z_2((y_1z_1)z_2)))(\lambda x_3y_3z_3.x_3z_3(y_3z_3))(\lambda x_4y_4.x_4)(\lambda x_5y_5.x_5)$

$\to_\beta (\lambda z_1.(\lambda z_2.z_1z_2(((\lambda x_3y_3z_3.x_3z_3(y_3z_3))z_1)z_2)))(\lambda x_4y_4.x_4)(\lambda x_5y_5.x_5)$

$\twoheadrightarrow_\beta (\lambda z_1.(\lambda z_2.z_1z_2(\lambda z_3.z_1z_3(z_2z_3))))(\lambda x_4y_4.x_4)(\lambda x_5y_5.x_5)$

$\twoheadrightarrow_\beta (\lambda x_4y_4.x_4)(\lambda x_5y_5.x_5)(\lambda z_3.z_1z_3(z_2z_3))$

$\twoheadrightarrow_\beta (\lambda x_5y_5.x_5) =_\alpha K$

$(\lambda x_1y_1z_1.x_1z_1(y_1z_1))(\lambda x_2y_2.x_2)(\lambda x_3y_3.x_3)(\lambda x_4y_4.x_4)$

$\twoheadrightarrow_\beta (\lambda z_1.(\lambda x_2y_2.x_2)z_1((\lambda x_3y_3.x_3)z_1))(\lambda x_4y_4.x_4)$

$\twoheadrightarrow_\beta (\lambda z_1.(\lambda y_2.z_1)(\lambda y_3.z_1))(\lambda x_4y_4.x_4)$

$\to_\beta (\lambda y_2.(\lambda x_4y_4.x_4))(\lambda y_3.(\lambda x_4y_4.x_4))$

$\to_\beta (\lambda x_4y_4.x_4) =_\alpha K$ ✓

1.10) zero $:= \lambda fx.x$
one $:= \lambda fx.fx$
two $:= \lambda fx.f(fx)$
add $:= \lambda mnfx.mf(nfx)$
mult $:= \lambda mnfx.m(nf)x$

(a) add one one $\twoheadrightarrow_\beta$ two

$(\lambda mnfx.mf(nfx))(\lambda fx.fx)(\lambda fx.fx) \twoheadrightarrow_\beta (\lambda fx.(\lambda fx.fx)f((\lambda fx.fx)fx))$

$\twoheadrightarrow_\beta \lambda fx.f(fx) \equiv$ two ✓

(b) add one one $\neq_\beta$ mult one zero

$(\lambda mnfx.m(nf)x)(\lambda fx.fx)(\lambda fx.x) \twoheadrightarrow_\beta (\lambda fx.(\lambda fx.fx)((\lambda fx.x)f)x)$

$\twoheadrightarrow_\beta \lambda fx.((\lambda fx.x)f)x \to_\beta \lambda fx.x \equiv$ zero $\neq_\beta$ two ✓

1.11) suc $:= \lambda mfx.f(mfx)$

(a) suc zero $=_\beta$ one

$(\lambda mfx.f(mfx))(\lambda fx.x) \to_\beta \lambda fx.f((\lambda fx.x)fx) \twoheadrightarrow_\beta \lambda fx.fx \equiv$ one ✓

(b) suc one $=_\beta$ two

$(\lambda mfx.f(mfx))(\lambda fx.fx) \to_\beta \lambda fx.f((\lambda fx.fx)fx) \twoheadrightarrow_\beta \lambda fx.f(fx) \equiv$ two ✓

1.12) true $:= K = \lambda xy.x$
false $:=$ zero $= \lambda xy.y$
not $:= \lambda z.z$ false true
$p \in \Lambda$

(a) $p \twoheadrightarrow_\beta$ true $\Rightarrow$ not(not $p$) $=_\beta p$

not(not $p$) $\twoheadrightarrow_\beta$ not(not true) $\equiv (\lambda z.z(\lambda xy.y)(\lambda xy.x))((\lambda z.z(\lambda xy.y)(\lambda xy.x))(\lambda xy.x))$

$\to_\beta (\lambda z.z(\lambda xy.y)(\lambda xy.x))((\lambda xy.x)(\lambda xy.y)(\lambda xy.x))$

$\twoheadrightarrow_\beta (\lambda xy.y)(\lambda xy.y)(\lambda xy.x) \twoheadrightarrow_\beta (\lambda xy.x) \equiv$ true $\twoheadleftarrow_\beta p$ ✓

(b) $p \twoheadrightarrow_\beta$ false $\Rightarrow$ not(not $p$) $=_\beta p$

not(not $p$) $\twoheadrightarrow_\beta$ not(not false) $\equiv (\lambda z.z(\lambda xy.y)(\lambda xy.x))(\lambda z.z(\lambda xy.y)(\lambda xy.x))(\lambda xy.y)$

$\twoheadrightarrow_\beta (\lambda xy.y)(\lambda xy.y)(\lambda xy.x)(\lambda xy.y)(\lambda xy.x) \twoheadrightarrow_\beta (\lambda xy.x)(\lambda xy.y)(\lambda xy.x) \twoheadrightarrow_\beta$

$\twoheadrightarrow_\beta (\lambda xy.y) \equiv$ false $\twoheadleftarrow_\beta p$ ✓

1.13) $iszero := \lambda z. z(\lambda x. false)\, true$

(a) $iszero\ zero \twoheadrightarrow_\beta true$

$(\lambda z. z(\lambda x. (\lambda xy.y))(\lambda xy.x))(\lambda f x. x) \to_\beta (\lambda f x. x)(\lambda x.(\lambda xy.y))(\lambda xy.x)$

$\twoheadrightarrow_\beta \lambda xy.x \equiv true$ ✓

(b) $iszero\ (\lambda f x.\ \underbrace{f(f(\ldots(x)))}_{n\text{-times},\ n \geq 1}) \twoheadrightarrow_\beta false$

$f^n(x) = f(f(\ldots(x)))$

$(\lambda z. z(\lambda x.(\lambda xy.y))(\lambda xy.x))(\lambda f x. f^n(x)) \to_\beta (\lambda f x. f^n(x))(\lambda x.(\lambda xy.y))(\lambda xy.x)$

$\twoheadrightarrow_\beta (\lambda x.(\lambda x.(\lambda xy.y))^n(x))(\lambda xy.x) \to_\beta (\lambda x.(\lambda xy.y))^n(x)$

$\equiv (\lambda x.(\lambda xy.y))^{n-1}((\lambda x.(\lambda xy.y))\ x) \to_\beta (\lambda x.(\lambda xy.y))^{n-1}(\lambda xy.y)$

$\twoheadrightarrow_\beta \lambda xy.y \equiv false$ ✓

1.14) "if $x$ then $u$ else $v$" $:= \lambda x. x u v$

(a) $\beta$-normal form of $(\lambda x. x u v)\, true = ?$

$(\lambda x. x u v)(\lambda xy.x) \to_\beta (\lambda xy.x) u v \twoheadrightarrow_\beta \underline{\underline{u}}$

(b) $\beta$-normal form of $(\lambda x. x u v)\, false = ?$

$(\lambda x. x u v)(\lambda xy.y) \to_\beta (\lambda xy.y) u v \twoheadrightarrow_\beta \underline{\underline{v}}$

1.15) $\Omega := (\lambda x.xx)(\lambda x.xx)$, $\Delta := \lambda x.xxx$

$\Omega$, $\Delta\Delta$ don't have $\beta$-normal forms

$$(\lambda x.xx)(\lambda x.xx) \to_\beta (\lambda x.xx)(\lambda x.xx)$$

$$(\lambda x.xxx)(\lambda x.xxx) \to_\beta (\lambda x.xxx)(\lambda x.xxx)(\lambda x.xxx)$$

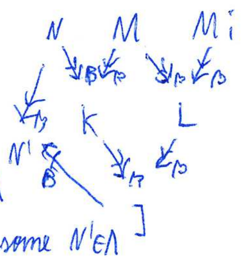
$\Omega$, $\Delta\Delta$ don't reduce to $\beta$-normal forms

Lemma 1.9.10 $\Rightarrow$ $\Omega$, $\Delta\Delta$ don't have $\beta$-normal forms ✓

1.16) $M \in \Lambda$ such that (1) $M$ has a $\beta$-normal form $N$

(2) There is a reduction path $M \equiv M_0 \to_\beta M_1 \to_\beta M_2 \to_\beta \ldots$ of infinite length

(a) $\forall i \in \mathbb{N}$: $M_i$ has a $\beta$-normal form




$M =_\beta M_i \overset{\text{corollary 1.9.9}}{\Rightarrow} M \twoheadrightarrow_\beta L,\ M_i \twoheadrightarrow_\beta L$ for some $L \in \Lambda$

(Lemma 1.9.10 $\Rightarrow M \twoheadrightarrow_\beta N$)

$M =_\beta N \Rightarrow M \twoheadrightarrow_\beta K,\ N \twoheadrightarrow_\beta K$ for some $K \in \Lambda$

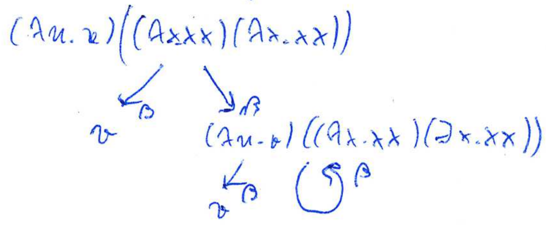
$K =_\beta L \Rightarrow K \twoheadrightarrow_\beta J,\ L \twoheadrightarrow_\beta J$ for some $J \in \Lambda$

$\Rightarrow M_i =_\beta N \Rightarrow M_i \twoheadrightarrow_\beta N',\ N \twoheadrightarrow_\beta N'$ for some $N' \in \Lambda$

Lemma 1.9.2 $\Rightarrow N \equiv N' \Rightarrow M_i \twoheadrightarrow_\beta N \Rightarrow N$ $\beta$ normal form of $M_i$ ✓

(b) Find example $M$

$M := (\lambda u. v)((\lambda x.xx)(\lambda x.xx))$



1.17) $MN$ strongly normalizing $\Rightarrow$ $M, N$ strongly normalizing

Suppose $M$ not strongly normalising $\Rightarrow$ $MN$ not s.n. ↯

Suppose $N$ not s.n $\Rightarrow$ $MN$ not s.n. ↯

1.18) $L, M, N \in \Lambda$, $L =_\beta M$, $L \twoheadrightarrow_\beta N$, $N$ in $\beta$-normal form

$M \twoheadrightarrow_\beta N$

$M =_\beta N \Rightarrow M \twoheadrightarrow_\beta K$, $N \twoheadrightarrow_\beta K$ for some $K \in \Lambda$

Lemma 1.9.2 $\Rightarrow N \equiv K \Rightarrow N \twoheadrightarrow_\beta N$ ✓

1.19) $U := \lambda zx. x(zzx)$

$Z := UU$

$Z$ is a fixed point combinator, i.e. $ZM$ fixed point for all $M \in \Lambda$, i.e. $M(ZM) = ZM$ for all $M \in \Lambda$

$M(ZM) \equiv M((\lambda zx. x(zzx)) \quad (\lambda zx. x(zzx))M) \to_\beta$

$$\left(\begin{array}{l} \to_\beta M((\lambda x. x((\lambda zx. x(zzx))(\lambda zx. x(zzx))x))M) \to_\beta M(M(\lambda zx. x(zzx))(\lambda zx. x(zzx))M) \\ \twoheadrightarrow_\beta M(M(x((\lambda zx. x(zzx))(\lambda zx. x(zzx))M))) \end{array}\right)$$

$ZM = (\lambda zx. x(zzx))(\lambda zx. x(zzx))M \twoheadrightarrow_\beta M((\lambda zx. x(zzx))(\lambda zx. x(zzx))M) \equiv M(ZM)$ ✓

1.20) (a) Find $M \in \Lambda$ such that $M =_\beta \lambda xy. xMy$

$L := \lambda z. (\lambda xy. xzy)$

Find $M \in \Lambda$ such that $M =_\beta LM \Rightarrow M =_\beta \lambda xy. xMy$

$M := (\lambda x. L(xx))(\lambda x. L(xx)) \Rightarrow \left(\begin{array}{l} LM =_\beta (\lambda x. L(xx))(\lambda x. L(xx))M \to_\beta \\ \to_\beta L((\lambda x. L(xx))(\lambda x. L(xx)))M \\ \to_\beta L \end{array}\right)$

$\Rightarrow M = (\lambda x. L(xx))(\lambda x. L(xx)) \to_\beta L((\lambda x. L(xx))(\lambda x. L(xx))) \equiv LM$ ✓

(b) Find $M \in \Lambda$ such that $Mxyz =_\beta xyzM$

Find $M \in \Lambda$ such that $M =_\beta \lambda xyz. xyzM \Rightarrow Mxyz =_\beta xyzM$

$L := \lambda m. (\lambda xyz. xyzm)$

Find $M \in \Lambda$ such that $M =_\beta LM \Rightarrow M =_\beta \lambda xyz. xyzM$

$M := (\lambda x. L(xx))(\lambda x. L(xx)) \Rightarrow M \to_\beta LM$ ✓