Recapitulation

Untyped lambda calculus:

xy , $\lambda x.xy$, $\lambda x.x =_\alpha \lambda y.y$, $(\lambda x.x)y \to_\beta y$

Typed lambda calculus, $\lambda\to$:

$$\frac{}{\Gamma \vdash x:\sigma} \text{ if } x:\sigma \in \Gamma \quad (var)$$

$$\frac{\Gamma \vdash M:\sigma\to\tau, \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau} \ (appl) \qquad \frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x:\sigma.M:\sigma\to\tau} \ (abst)$$

$\boxed{y:\alpha\to\beta}$

$\quad \boxed{z:\alpha}$

$\quad yz:\beta$

$\lambda z:\alpha.yz:\alpha\to\beta$

$\lambda y:\alpha\to\beta.\lambda z:\alpha.yz:(\alpha\to\beta)\to\alpha\to\beta$

Problems:

Well-typedness	:	$? \vdash term : ?$
Type assignment	:	$context \vdash term : ?$
Type checking	:	$context \overset{?}{\vdash} term : type$
Term finding	:	$context \vdash ? : type$

Terms depending on types, $\lambda 2$:

(var), (appl), (abst) as in $\lambda\to$,

$$\frac{}{\Gamma \vdash B:*} \text{ if } B \in \mathbb{T}_2 \text{, all free var in } B \text{ are declared in } \Gamma \quad (form)$$

$$\frac{\Gamma \vdash M:(\Pi\alpha:*.A), \quad \Gamma \vdash B:*}{\Gamma \vdash MB : A[\alpha := B]} \ (appl_2)$$

$$\frac{\Gamma, \alpha:* \vdash M:A}{\Gamma \vdash \lambda\alpha:*.M:\Pi\alpha:*.A} \ (abst_2)$$

Example:

$\boxed{\alpha:*}$

$\quad \boxed{f:\alpha\to\alpha}$

$\quad\quad \boxed{x:\alpha}$

$\quad\quad fx:\alpha$

$\quad\quad f(fx):\alpha$

$\quad \lambda x:\alpha.f(fx):\alpha\to\alpha$

$\lambda f:\alpha\to\alpha.\lambda x:\alpha.f(fx):(\alpha\to\alpha)\to\alpha\to\alpha$

$\lambda\alpha:*.\lambda f:\alpha\to\alpha.\lambda x:\alpha.f(fx):\Pi\alpha:*.(\alpha\to\alpha)\to\alpha\to\alpha$

Types dependent on types, $\lambda\underline{\omega}$:

$$\frac{}{\emptyset \vdash * : \square} \quad (\text{sort})$$

$$\frac{\Gamma \vdash A : s^{\swarrow \in \{*, \square\}}}{\Gamma, x : A \vdash x : A} \quad \text{if } x \notin \Gamma \quad (\text{var})$$

$$\frac{\Gamma \vdash A : B, \ \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \quad \text{if } x \notin \Gamma \quad (\text{weak})$$

$$\frac{\Gamma \vdash A : s, \ \Gamma \vdash B : s}{\Gamma \vdash A \to B : s} \quad (\text{form})$$

$$\frac{\Gamma \vdash M : A \to B, \ \Gamma \vdash N : A}{\Gamma \vdash MN : B} \quad (\text{appl})$$

$$\frac{\Gamma, x : A \vdash M : B, \ \Gamma \vdash A \to B : s}{\Gamma \vdash \lambda x : A . M : A \to B} \quad (\text{abst})$$

$$\frac{\Gamma \vdash A : B, \ \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \quad \text{if } B =_\beta B' \quad (\text{conv})$$

Example:

(1) $* : \square$ (sort)

$\boxed{\alpha : *}$

(2) $\alpha : *$ (var) on (1)

$\boxed{x : \alpha}$

(3) $x : \alpha$ (var) on (2)

(4) $\alpha : *$ (weak) on (2), ~~and~~ (1)

(5) $* : \square$ (weak) on (1), ~~and~~ (1)

$\boxed{\beta : *}$

(6) $\alpha : *$ (weak) on (2), ~~and~~ (2)

(7) $\beta : *$ (var) on (5)

(8) $\alpha \to \beta : *$ (form) on (6), (7)

(9) $* \to * : \square$ (form) on (5), (5)

$\boxed{\beta : *}$

(10) $* : \square$ (weak) on (1), (1)

$\boxed{\alpha : *}$

(11) $\alpha : *$ (var) on (10)

(12) $\alpha \to \alpha : *$ (form) on (11), (11)

(13) $* \to * : \square$ (form) on (10), (10)

(14) $\lambda \alpha : * . \alpha \to \alpha : * \to *$ (abst) on (12), (1)

(15) $\beta : *$ (var) on (1)

(16) $(\lambda \alpha : * . \alpha \to \alpha)\beta : *$ (appl) on (14), (15)

(17) $\beta \to \beta : *$ (form) on (15), (15)

$\boxed{x : (\lambda \alpha : * . \alpha \to \alpha)\beta}$

(18) $x : (\lambda \alpha : * . \alpha \to \alpha)\beta$ (var) on (16)

(19) $\beta \to \beta : *$ (form) on (15), (15)

(20) $x : \beta \to \beta$ (conv) on (18), (19)

Shortened derivations:

Silently execute: (sort)
(var)
(weak)
(form)
second premiss of (abst)

(a) $\boxed{\beta : *}$

(b) $\boxed{\alpha : *}$

(12) $\alpha \to \alpha : *$

necessary for (14) $\downarrow$ (form) on (b), (b)

(14) $\lambda \alpha : * . \alpha \to \alpha : * \to *$ (abst) on (12) ← first premiss only

(16) $(\lambda \alpha : * . \alpha \to \alpha)\beta : *$ (appl) on (14), (a)