p-adic systems - a survey

Results from "An introduction to p-adic systems: A new kind of number system inspired by the Collatz $3n+1$ conjecture" (in preparation)

Mario Weitzer

Graz University of Technology, Austria

Advanced topics in discrete mathematics

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p-fibred function: piecewise function on $\mathbb{Z}_p$, branches for all residue classes mod p

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$$F : \mathbb{Z}_p \to \mathbb{Z}_p$$

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where

- $2 \leq p \in \mathbb{N}$ (not necessarily prime)
- $F[0], \ldots, F[p-1] : \mathbb{Z}_p \to \mathbb{Z}_p$

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Example: $F_C = (x, 3x+1)$

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Apply F_C iteratively:

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Example: $F_C = (x, 3x + 1)$

Apply F_C iteratively: 17

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Example: $F_C = (x, 3x+1)$

Apply F_C iteratively: $17 \xrightarrow{F_C} 26$

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Example: $F_C = (x, 3x+1)$

Apply F_C iteratively: $17 \xrightarrow{F_C} 26 \xrightarrow{F_C} 13$

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Apply F_C iteratively: $17 \xrightarrow{F_C} 26 \xrightarrow{F_C} 13 \xrightarrow{F_C} 20 \xrightarrow{F_C} 10 \xrightarrow{F_C} 5$

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Apply F_C iteratively: $17 \xrightarrow{F_C} 26 \xrightarrow{F_C} 13 \xrightarrow{F_C} 20 \xrightarrow{F_C} 10 \xrightarrow{F_C} 5 \xrightarrow{F_C} 8$

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$\xrightarrow{F_C} 4 \xrightarrow{F_C} 2$

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Collatz conjecture: All orbits of $n \in \mathbb{N}$ under F_C end up in the cycle $(1, 2)$

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Collatz conjecture: All orbits of $n \in \mathbb{N}$ under F_C end up in the cycle $(1, 2)$

Notation: $S(F_C)[17] = (17, 26, 13, 20, 10, \ldots)$: F_C-sequence of 17

Definitions

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Notation: $S(F_C)[17] = (17, 26, 13, 20, 10, \ldots)$: F_C-sequence of 17
$D(F_C)[17] = (1, 0, 1, 0, 0, \ldots)$: F_C-(digit) expansion of 17

What do $F_C = (x, 3x+1)$ and $F_2 = (x, x-1)$ have in common?

Tables of sequences:

1	1	0	0	0	$\cdots$
2	2	1	0	0	$\cdots$
3	3	1	0	0	$\cdots$
4	4	2	1	0	$\cdots$
5	5	2	1	0	$\cdots$
6	6	3	1	0	$\cdots$
7	7	3	1	0	$\cdots$
8	8	4	2	1	$\cdots$
9	9	4	2	1	$\cdots$
10	10	5	2	1	$\cdots$
11	11	5	2	1	$\cdots$
12	12	6	3	1	$\cdots$
13	13	6	3	1	$\cdots$
14	14	7	3	1	$\cdots$
15	15	7	3	1	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$S(F_2)$	0	1	2	3	$\cdots$

1	1	2	1	2	$\cdots$
2	2	1	2	1	$\cdots$
3	3	5	8	4	$\cdots$
4	4	2	1	2	$\cdots$
5	5	8	4	2	$\cdots$
6	6	3	5	8	$\cdots$
7	7	11	17	26	$\cdots$
8	8	4	2	1	$\cdots$
9	9	14	7	11	$\cdots$
10	10	5	8	4	$\cdots$
11	11	17	26	13	$\cdots$
12	12	6	3	5	$\cdots$
13	13	20	10	5	$\cdots$
14	14	7	11	17	$\cdots$
15	15	23	35	53	$\cdots$
16	16	8	4	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$S(F_C)$	0	1	2	3	$\cdots$

Definitions

What do $F_C = (x, 3x+1)$ and $F_2 = (x, x-1)$ have in common?

Tables of expansions:

1	1	0	0	0	$\cdots$
2	0	1	0	0	$\cdots$
3	1	1	0	0	$\cdots$
4	0	0	1	0	$\cdots$
5	1	0	1	0	$\cdots$
6	0	1	1	0	$\cdots$
7	1	1	1	0	$\cdots$
8	0	0	0	1	$\cdots$
9	1	0	0	1	$\cdots$
10	0	1	0	1	$\cdots$
11	1	1	0	1	$\cdots$
12	0	0	1	1	$\cdots$
13	1	0	1	1	$\cdots$
14	0	1	1	1	$\cdots$
15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$D(F_2)$	0	1	2	3	$\cdots$

1	1	0	1	0	$\cdots$
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4	0	0	1	0	$\cdots$
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6	0	1	1	0	$\cdots$
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10	0	1	0	0	$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$D(F_C)$	0	1	2	3	$\cdots$

What do $F_C = (x, 3x+1)$ and $F_2 = (x, x-1)$ have in common?

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6	0	1	1	0	$\cdots$
7	1	1	1	0	$\cdots$
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10	0	1	0	1	$\cdots$
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13	1	0	0	1	$\cdots$
14	0	1	1	1	$\cdots$
15	1	1	1	1	$\cdots$
16	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$D(F_C)$	0	1	2	3	$\cdots$

What do $F_C = (x, 3x+1)$ and $F_2 = (x, x-1)$ have in common?

Tables of expansions:

1	1	0	0	0	⋯
2	0	1	0	0	⋯
3	1	1	0	0	⋯
4	0	0	1	0	⋯
5	1	0	1	0	⋯
6	0	1	1	0	⋯
7	1	1	1	0	⋯
8	0	0	0	1	⋯
9	1	0	0	1	⋯
10	0	1	0	1	⋯
11	1	1	0	1	⋯
12	0	0	1	1	⋯
13	1	0	1	1	⋯
14	0	1	1	1	⋯
15	1	1	1	1	⋯
16	0	0	0	0	⋯
⋮	⋮	⋮	⋮	⋮	⋱
$D(F_2)$	0	1	2	3	⋯

1	1	0	1	0	⋯
2	0	1	0	1	⋯
3	1	1	0	0	⋯
4	0	0	1	0	⋯
5	1	0	0	0	⋯
6	0	1	1	0	⋯
7	1	1	1	0	⋯
8	0	0	0	1	⋯
9	1	0	1	1	⋯
10	0	1	0	0	⋯
11	1	1	0	1	⋯
12	0	0	1	1	⋯
13	1	0	0	1	⋯
14	0	1	1	1	⋯
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⋮	⋮	⋮	⋮	⋮	⋱
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First k digits of expansions of m and n coincide $\Leftrightarrow$ $m \equiv n \mod 2^k$ (Block property)

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Definition

p-fibred function:
- piecewise function on $\mathbb{Z}_p$
- branches for all residue classes mod p

p-adic system:
- block property

p-adic systems

p-adic systems

$F = (F[0], \ldots, F[p-1])$ (p-fibred function)

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1	1	0	1	0	1	0	1	0	…
2	0	1	0	1	0	1	0	1	…
3	1	1	0	0	0	1	0	1	…
4	0	0	1	0	1	0	1	0	…
5	1	0	0	0	1	0	1	0	…
6	0	1	1	0	0	0	1	0	…
7	1	1	1	0	1	0	0	1	…
8	0	0	0	1	0	1	0	1	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
D	0	1	2	3	4	5	6	7	…

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4	0	0	1	0	1	0	1	0	…
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7	1	1	1	0	1	0	0	1	…
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⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

p-adic permutations

Three and a half interpretations

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4	0	0	1	0	1	0	1	0	…
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$\pi_{F,G} := {\psi_G}^{-1} \circ \psi_F$

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7	1	1	1	0	1	0	0	1	…
8	0	0	0	1	0	1	0	1	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
D	0	1	2	3	4	5	6	7	…

p-adic permutations

$\pi : \mathbb{Z}_p \to \mathbb{Z}_p$

$\pi(n) \equiv n \mod p$

$m \equiv n \mod p^k \Leftrightarrow \pi(m) \equiv \pi(n) \mod p^k$

for all $k \in \mathbb{N}$ and $m, n \in \mathbb{Z}_p$

$\psi_F : \mathbb{Z}_p \to \underline{p}^{\mathbb{N}_0}$, $\psi_F(n) = D(F)[n]$ (bijective)

$\pi_{F,G} := {\psi_G}^{-1} \circ \psi_F$

$\pi := \pi_{(x,3x+1),(x,x-1)}$.

p-adic systems

$F = (F[0], \dots, F[p-1]) \quad (p\text{-fibred function})$

$m \equiv n \mod p^k \Leftrightarrow F[r](m) \equiv F[r](n) \mod p^k$

for all $k \in \mathbb{N}$ and $m, n \in r + p\mathbb{Z}_p$

$(\Leftrightarrow F[r]$ (p, r)-suitable for all $r)$

$F = (x, 3x + 1)$

ordinary functions with block property

$f : \mathbb{Z}_p \to \mathbb{Z}_p$

$m \equiv n \mod p^k \Leftrightarrow f(m) \equiv f(n) \mod p^{k-1}$

for all $k \in \mathbb{N}$ and $m, n \in \mathbb{Z}_p$ with $m \equiv n \mod p$

$f = \left(x \text{ even ? } \frac{x}{2} : \frac{3x+1}{2}\right)$

p-digit tables with block property

$D \in \underline{p}^{\mathbb{Z}_p \times \mathbb{N}_z} \quad (\underline{x} := \{0, \dots, x-1\})$

$D[n][0] = n \,\%\, p$

$m \equiv n \mod p^k \Leftrightarrow D[m][\underline{k}] = D[n][\underline{k}]$

for all $k \in \mathbb{N}$ and $m, n \in \mathbb{Z}_p$

1	1	0	1	0	1	0	1	0	…
2	0	1	0	1	0	1	0	1	…
3	1	1	0	0	0	1	0	1	…
4	0	0	1	0	1	0	1	0	…
5	1	0	0	0	1	0	1	0	…
6	0	1	1	0	0	0	1	0	…
7	1	1	1	0	1	0	0	1	…
8	0	0	0	1	0	1	0	1	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
D	0	1	2	3	4	5	6	7	…

p-adic permutations

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for all $k \in \mathbb{N}$ and $m, n \in \mathbb{Z}_p$

$\psi_F : \mathbb{Z}_p \to \underline{p}^{\mathbb{N}_0}, \ \psi_F(n) = D(F)[n] \quad \text{(bijective)}$

$\pi_{F,G} := {\psi_G}^{-1} \circ \psi_F$

$\pi := \pi_{(x,3x+1),(x,x-1)}$:

$\pi(0) = 0, \ \pi(1) = -1/3, \ \pi(2) = -2/3$

Three and a half interpretations

p-adic systems

$F = (F[0], \dots, F[p-1])$ (p-fibred function)

$m \equiv n \mod p^k \Leftrightarrow F[r](m) \equiv F[r](n) \mod p^k$

for all $k \in \mathbb{N}$ and $m, n \in r + p\mathbb{Z}_p$

($\Leftrightarrow$ $F[r]$ (p, r)-suitable for all r)

$F = (x, 3x+1)$

ordinary functions with block property

$f : \mathbb{Z}_p \to \mathbb{Z}_p$

$m \equiv n \mod p^k \Leftrightarrow f(m) \equiv f(n) \mod p^{k-1}$

for all $k \in \mathbb{N}$ and $m, n \in \mathbb{Z}_p$ with $m \equiv n \mod p$

$f = \left(x \text{ even } ? \ \frac{x}{2} : \frac{3x+1}{2}\right)$

p-digit tables with block property

$D \in \underline{p}^{\mathbb{Z}_p \times \mathbb{N}_z}$ $(\underline{x} := \{0, \dots, x-1\})$

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for all $k \in \mathbb{N}$ and $m, n \in \mathbb{Z}_p$

1	1	0	1	0	1	0	1	0	…
2	0	1	0	1	0	1	0	1	…
3	1	1	0	0	0	1	0	1	…
4	0	0	1	0	1	0	1	0	…
5	1	0	0	0	1	0	1	0	…
6	0	1	1	0	0	0	1	0	…
7	1	1	1	0	1	0	0	1	…
8	0	0	0	1	0	1	0	1	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
D	0	1	2	3	4	5	6	7	…

p-adic permutations

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$\pi(0) = 0$, $\pi(1) = -1/3$, $\pi(2) = -2/3$

$\pi(17) = -401/3$, $\pi(-19/9) = 17$,

p-adic systems

$F = (F[0], \dots, F[p-1])$ (p-fibred function)

$m \equiv n \mod p^k \Leftrightarrow F[r](m) \equiv F[r](n) \mod p^k$

for all $k \in \mathbb{N}$ and $m, n \in r + p\mathbb{Z}_p$

($\Leftrightarrow$ $F[r]$ (p, r)-suitable for all r)

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ordinary functions with block property

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$m \equiv n \mod p^k \Leftrightarrow f(m) \equiv f(n) \mod p^{k-1}$

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for all $k \in \mathbb{N}$ and $m, n \in \mathbb{Z}_p$

1	1	0	1	0	1	0	1	0	…
2	0	1	0	1	0	1	0	1	…
3	1	1	0	0	0	1	0	1	…
4	0	0	1	0	1	0	1	0	…
5	1	0	0	0	1	0	1	0	…
6	0	1	1	0	0	0	1	0	…
7	1	1	1	0	1	0	0	1	…
8	0	0	0	1	0	1	0	1	…
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$\pi(0) = 0$, $\pi(1) = -1/3$, $\pi(2) = -2/3$

$\pi(17) = -401/3$, $\pi(-19/9) = 17$,

$\pi(-23/31) = 975575385/8388607$

$f\ (p,r)$-suitable $\quad\Leftrightarrow\quad \forall\, k \in \mathbb{N} : \forall\, m,n \in r+p\mathbb{Z}_p : \big(m \equiv n\ \ (p^k) \Leftrightarrow f(m) \equiv f(n)\ \ (p^k)\big)$

f (p,r)-suitable $\Leftrightarrow$ $\forall\, k \in \mathbb{N} : \forall\, m, n \in r + p\mathbb{Z}_p : \big(m \equiv n \;\; (p^k) \Leftrightarrow f(m) \equiv f(n) \;\; (p^k)\big)$

$\mathbb{Z}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Z}_p[x]$ for all $r \in \underline{p}$

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- $(x)^p$, $(x, 3x+1)$
- $\left(7x^3 - 4x^2 + x - 6, 3x^7 - x + 1, x^2 + 6x + 2\right)$

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- $\big(\mathrm{i}x^2 + x, 5\mathrm{i}x^4 - 2 + 7, x + 3, -9x^3 + 12x + 7, -5\mathrm{i}x^2 + x + 1\big)$
  where $\mathrm{i}^2 = -1$, i.e. $\mathrm{i} = \ldots 2431212$ or $\mathrm{i} = \ldots 2013233$

f (p,r)-suitable $\Leftrightarrow$ $\forall\, k \in \mathbb{N} : \forall\, m,n \in r + p\mathbb{Z}_p : \big(m \equiv n \ (p^k) \Leftrightarrow f(m) \equiv f(n) \ (p^k)\big)$

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- $\big(\prod_{i=0}^{p-1}(x-i)\big)^p$ where $p \in \mathbb{P}$

Examples: p-adic systems

f (p,r)-suitable $\quad\Leftrightarrow\quad \forall\, k \in \mathbb{N} : \forall\, m,n \in r+p\mathbb{Z}_p : \big(m \equiv n \ \ (p^k) \Leftrightarrow f(m) \equiv f(n) \ \ (p^k)\big)$

$\mathbb{Z}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Z}_p[x]$ for all $r \in \underline{p}$

$F[r]$ (p,r)-suitable $\quad\Leftrightarrow\quad \gcd\left(p, F[r]'(r)\right) = 1$

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- $\big(7x^3 - 4x^2 + x - 6, 3x^7 - x + 1, x^2 + 6x + 2\big)$
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- $\big(\mathrm{i}x^2 + x, 5\mathrm{i}x^4 - 2 + 7, x + 3, -9x^3 + 12x + 7, -5\mathrm{i}x^2 + x + 1\big)$
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- $\left(\prod_{i=0}^{p-1}(x-i)\right)^p$ where $p \in \mathbb{P}$

$\mathbb{Q}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Q}_p[x]$ for all $r \in \underline{p}$

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- $\big(\mathrm{i}x^2 + x, 5\mathrm{i}x^4 - 2 + 7, x + 3, -9x^3 + 12x + 7, -5\mathrm{i}x^2 + x + 1\big)$
  where $\mathrm{i}^2 = -1$, i.e. $\mathrm{i} = \ldots 2431212$ or $\mathrm{i} = \ldots 2013233$
- $\big(\prod_{i=0}^{p-1}(x - i)\big)^p$ where $p \in \mathbb{P}$

$\mathbb{Q}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Q}_p[x]$ for all $r \in \underline{p}$

$F[r]\ (p,r)$-suitable $\quad\Leftrightarrow\quad$ $F[r]\ (p,r)$-suitable on a computable finite witness set

$f\ (p,r)$-suitable $\quad\Leftrightarrow\quad \forall\, k \in \mathbb{N} : \forall\, m,n \in r + p\mathbb{Z}_p : \big(m \equiv n \ (p^k) \Leftrightarrow f(m) \equiv f(n)\ (p^k)\big)$

$\mathbb{Z}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Z}_p[x]$ for all $r \in \underline{p}$

$F[r]\ (p,r)$-suitable $\quad\Leftrightarrow\quad \gcd(p, F[r]'(r)) = 1$

- $(x)^p$, $(x, 3x+1)$
- $\big(7x^3 - 4x^2 + x - 6, 3x^7 - x + 1, x^2 + 6x + 2\big)$
- $\left(\frac{32}{7}x^2 + \frac{5}{3}x - 4, \frac{13}{11}x + 5, \frac{1}{17}x + 2, 3x^2 + \frac{7}{19}x - \frac{14}{5}\right)$
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$\mathbb{Q}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Q}_p[x]$ for all $r \in \underline{p}$

$F[r]\ (p,r)$-suitable $\quad\Leftrightarrow\quad$ $F[r]\ (p,r)$-suitable on a computable finite witness set

- $\left(\frac{17}{4}x^6 + \frac{37}{16}x^5 - \frac{15}{4}x^3 + 3x - 2, -25x^6 + \frac{7}{4}x^5 - \frac{49}{2}x^4 - \frac{21}{2}x^3 + 5x^2 + \frac{79}{4}x + \frac{19}{2}\right)$

f (p,r)-suitable $\Leftrightarrow$ $\forall\, k \in \mathbb{N} : \forall\, m,n \in r+p\mathbb{Z}_p : \left(m \equiv n \ (p^k) \Leftrightarrow f(m) \equiv f(n) \ (p^k)\right)$

$\mathbb{Z}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Z}_p[x]$ for all $r \in \underline{p}$

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- $\left(7x^3 - 4x^2 + x - 6, 3x^7 - x + 1, x^2 + 6x + 2\right)$
- $\left(\frac{32}{7}x^2 + \frac{5}{3}x - 4, \frac{13}{11}x + 5, \frac{1}{17}x + 2, 3x^2 + \frac{7}{19}x - \frac{14}{5}\right)$
- $\left(\mathrm{i}x^2 + x, 5\mathrm{i}x^4 - 2 + 7, x + 3, -9x^3 + 12x + 7, -5\mathrm{i}x^2 + x + 1\right)$
  where $\mathrm{i}^2 = -1$, i.e. $\mathrm{i} = \ldots 2431212$ or $\mathrm{i} = \ldots 2013233$
- $\left(\prod_{i=0}^{p-1}(x-i)\right)^p$ where $p \in \mathbb{P}$

$\mathbb{Q}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Q}_p[x]$ for all $r \in \underline{p}$

$F[r]$ (p,r)-suitable $\Leftrightarrow$ $F[r]$ (p,r)-suitable on a computable finite witness set

- $\left(\frac{17}{4}x^6 + \frac{37}{16}x^5 - \frac{15}{4}x^3 + 3x - 2, -25x^6 + \frac{7}{4}x^5 - \frac{49}{2}x^4 - \frac{21}{2}x^3 + 5x^2 + \frac{79}{4}x + \frac{19}{2}\right)$
- $\left(-\frac{23}{27}x^3 + x - 2, -28x^4 + \frac{7}{9}x^3 + \frac{29}{3}x^2 + \frac{4}{3}x + \frac{11}{9}, -2x^4 - \frac{29}{9}x^3 - \frac{5}{3}x^2 - \frac{11}{3}x - \frac{2}{9}\right)$

$f\ (p,r)$-suitable $\quad\Leftrightarrow\quad \forall\, k \in \mathbb{N} : \forall\, m,n \in r + p\mathbb{Z}_p : \big(m \equiv n\ \ (p^k) \Leftrightarrow f(m) \equiv f(n)\ \ (p^k)\big)$

$\mathbb{Z}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Z}_p[x]$ for all $r \in \underline{p}$

$F[r]\ (p,r)$-suitable $\quad\Leftrightarrow\quad \gcd\left(p, F[r]'(r)\right) = 1$

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- $\big(7x^3 - 4x^2 + x - 6, 3x^7 - x + 1, x^2 + 6x + 2\big)$
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- $\big(\mathrm{i}x^2 + x, 5\mathrm{i}x^4 - 2 + 7, x + 3, -9x^3 + 12x + 7, -5\mathrm{i}x^2 + x + 1\big)$
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- $\big(-\frac{23}{27}x^3 + x - 2, -28x^4 + \frac{7}{9}x^3 + \frac{29}{3}x^2 + \frac{4}{3}x + \frac{11}{9}, -2x^4 - \frac{29}{9}x^3 - \frac{5}{3}x^2 - \frac{11}{3}x - \frac{2}{9}\big)$

$\mathbb{Z}_p$-rational p-adic systems: $F[r] = P_r/Q_r$ with $P_r(x), Q_r(x) \in \mathbb{Z}_p[x]$ for all $r \in \underline{p}$

f (p,r)-suitable $\Leftrightarrow$ $\forall\, k \in \mathbb{N} : \forall\, m,n \in r + p\mathbb{Z}_p : \big(m \equiv n \;\; (p^k) \Leftrightarrow f(m) \equiv f(n) \;\; (p^k)\big)$

$\mathbb{Z}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Z}_p[x]$ for all $r \in \underline{p}$

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- $\left(\mathrm{i}x^2 + x, 5\mathrm{i}x^4 - 2 + 7, x + 3, -9x^3 + 12x + 7, -5\mathrm{i}x^2 + x + 1\right)$
  where $\mathrm{i}^2 = -1$, i.e. $\mathrm{i} = \ldots 2431212$ or $\mathrm{i} = \ldots 2013233$
- $\left(\prod_{i=0}^{p-1}(x - i)\right)^p$ where $p \in \mathbb{P}$

$\mathbb{Q}_p$-polynomial p-adic systems: $F[r](x) \in \mathbb{Q}_p[x]$ for all $r \in \underline{p}$

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Examples: p-adic systems

$f\ (p,r)$-suitable $\Leftrightarrow$ $\forall\, k \in \mathbb{N} : \forall\, m,n \in r+p\mathbb{Z}_p : \big(m \equiv n\ (p^k) \Leftrightarrow f(m) \equiv f(n)\ (p^k)\big)$

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Thue-Morse sequence: $T = (0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots)$
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D, E the 2-digit tables:
(R, S = concatenation of column periods of D, E)

0	0	1	1	$\cdots$
1	1	0	0	$\cdots$
2	0	0	0	$\cdots$
3	1	1	1	$\cdots$
4	0	1	0	$\cdots$
5	1	0	1	$\cdots$
6	0	0	1	$\cdots$
7	1	1	0	$\cdots$
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7	1	1	0	…
⋮	⋮	⋮	⋮	⋱

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0	0	0	0	…
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4	0	1	1	…
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D, E have the block property

Examples: p-digit tables with block property

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$D = D(F)$ for $F := (x + 6 - 2(x \,\%\, 8), x + 3 - 2(x \,\%\, 8) + 2(x \,\%\, 4))$

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p-permutation polynomial: $f \in \mathbb{Z}_p[x]$

$$f_k : \mathbb{Z}_p/p^k\mathbb{Z}_p \to \mathbb{Z}_p/p^k\mathbb{Z}_p \text{ bijective for all } k \in \mathbb{N} \ (k=2)$$
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  $F = \left(\frac{\sqrt{8x^2-28x+22+3}+7}{2}, \frac{\sqrt{3(8x^2-28x+22)+3}+7}{2} \right)$, $G = (x, 3x+1)$

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Applications of p-adic systems: generalizing Hensel's Lemma

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Example:
$$F[0](x) := 7x^3 - 4x^2 + x - 6$$
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for a unique $n \in \mathbb{Z}_3$

Applications of p-adic systems: generalizing the Collatz conjecture

$F_C = (x, 3x + 1)$

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Known periods on $\mathbb{Z}$

Digit period ($D(F_C)$)	Sequence period ($S(F_C)$)
0	0
$1, 0$	$1, 2$
1	-1
$1, 1, 0$	$-5, -7, -10$
$1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0$	$-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34$

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- Do expansions of the integers admit other periods?
- Are there rational numbers that have aperiodic expansions?

Definition: $n \in \mathbb{Z}_p$ periodic point of p-adic system F: $D(F)[n]$ ultimately periodic
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- $((p + 1)x, x, \ldots, x)$

Mathematica!

Applications of p-adic permutations: generalized Collatz conjecture

Theorem: If $F = (a_0' + b_0 x, a_1' + b_1 x)$ and $G = (a_0'' + b_0 x, a_1'' + b_1 x)$, then:
$\pi_{F,G}(n) = \frac{a_0' a_1'' - a_1' a_0'' + n((b_0 - 2)a_1'' - (b_1 - 2)a_0'')}{(b_0 - 2)a_1' - (b_1 - 2)a_0'}$ for all $n \in \mathbb{Z}_2$

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Corollary: If $a_0', a_1', a_0'', a_1'', b_0, b_1 \in \mathbb{Q} \cap \mathbb{Z}_2$, then
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$\pi_{F,G}(n)$ "$=$" $\frac{1+n(b_1-2)}{2-b_0}$ for all $n \in \mathbb{Z}_2$ ("$=$": small technical detail)

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$\pi_{F,G}(n) = \frac{a'_0 a''_1 - a'_1 a''_0 + n((b_0-2)a''_1 - (b_1-2)a''_0)}{(b_0-2)a'_1 - (b_1-2)a'_0}$ for all $n \in \mathbb{Z}_2$

Corollary: If $a'_0, a'_1, a''_0, a''_1, b_0, b_1 \in \mathbb{Q} \cap \mathbb{Z}_2$, then
$\pi_{F,G}(\mathbb{Q} \cap \mathbb{Z}_2) = \mathbb{Q} \cap \mathbb{Z}_2$ and thus $\mathrm{PerP}(F) = \mathbb{Q} \cap \mathbb{Z}_2 \Leftrightarrow \mathrm{PerP}(G) = \mathbb{Q} \cap \mathbb{Z}_2$

The constant coefficients are irrelevant!

Theorem: If $F = (b_0x, 1 + b_1x)$ and $G = (b_1x, 1 + b_0x)$, then:
$\pi_{F,G}(n)$ "$=$" $\frac{1+n(b_1-2)}{2-b_0}$ for all $n \in \mathbb{Z}_2$ ("$=$": small technical detail)

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If $m := \frac{156065447}{59288775}$, $n := -\frac{847767822}{612650675}$, then $\frac{-35-3m}{31} = n$ and
$F_1(m) = (1, 0, 1, 0) \cdot (1, 1, 0, 1, 0, 0, 0, 1)^\infty$
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Applications of p-adic permutations: group structure on p-adic systems

Fix p-adic system G and define group structure on set of all p-adic systems by:
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More examples of p-adic systems!

Definition: Tree of cycles $(\mathcal{G}(\pi), c(\pi))$ of p-adic permutation π by example

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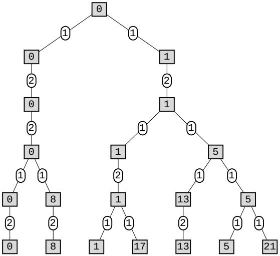

Applications of p-adic permutations: tree of cycles

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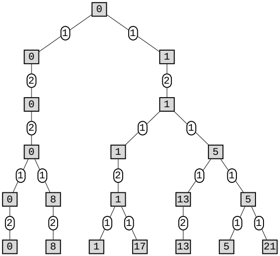
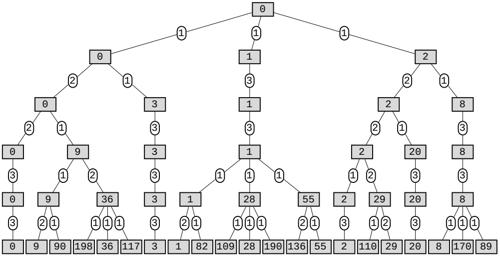
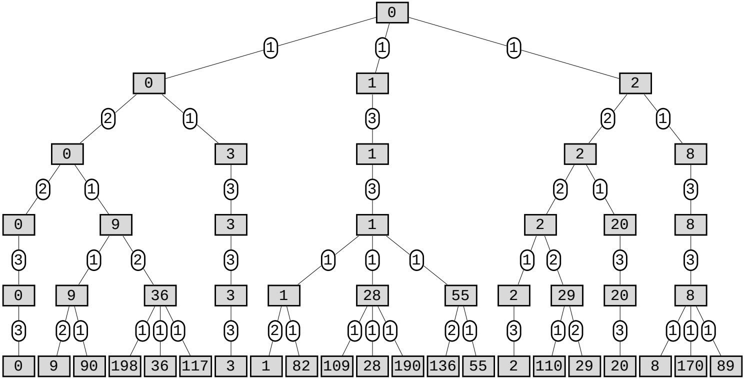


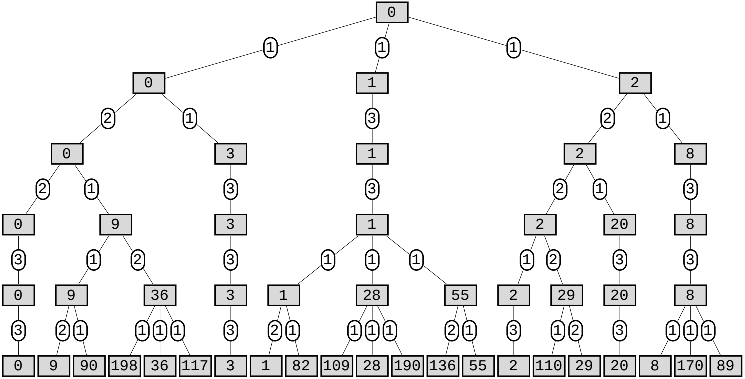


$(\pi \;:=\; \pi_{(-5x-3,5x+1,-x+5),(-4x+3,-x+1,-2x+4)})$

Applications of p-adic permutations: tree of cycles



Theorem: $\mathcal{G}(\pi)$ is a directed, infinite, rooted tree with root $(0,(0))$.



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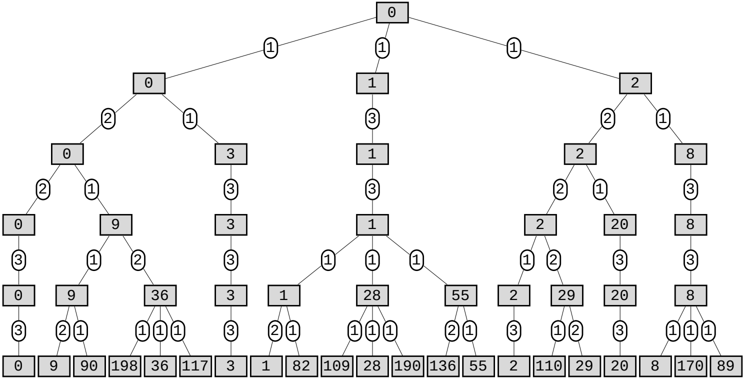

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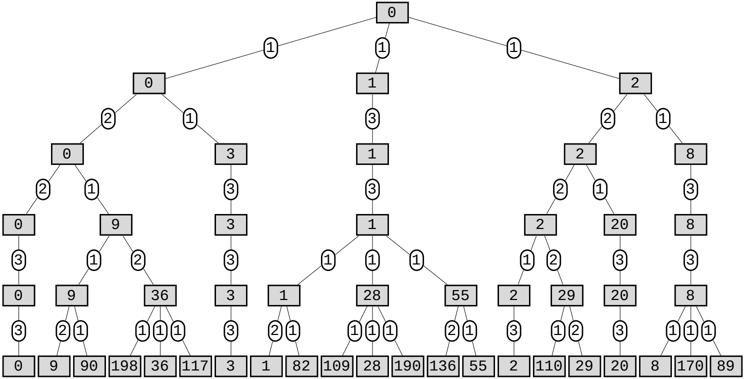

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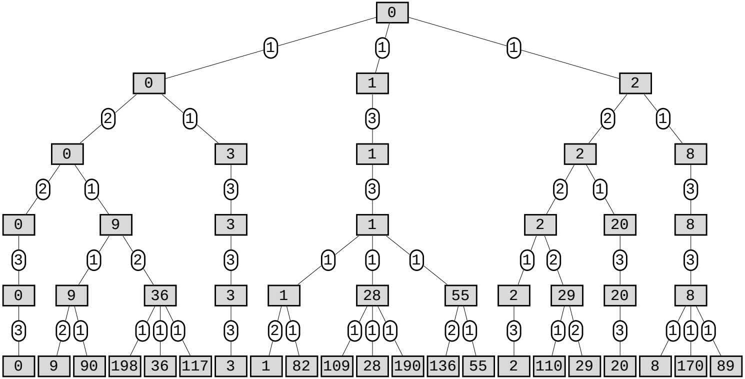

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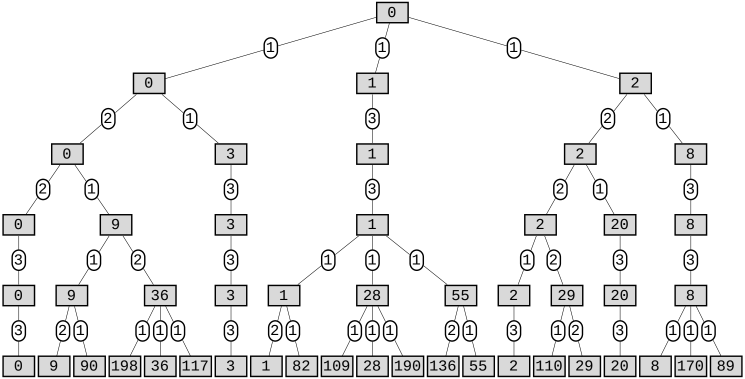
Applications of p-adic permutations: tree of cycles



Theorem: $\mathcal{G}(\pi)$ is a directed, infinite, rooted tree with root $(0,(0))$. The out-degrees of all vertices are contained in $\{1,\dots,p\}$ and the out-degree of the root is p. The cycle decomposition of π_k is given by the k-th layer of $\mathcal{G}(\pi)$ (vertices of distance k from the root). $c(\pi)$ defines an edge labeling of $\mathcal{G}(\pi)$. The labels of all outgoing edges of a given vertex sum up to p and all edges going out of the root are labeled 1. For every vertex of $\mathcal{G}(\pi)$ the length of the represented cycle coincides with the product of all edge labels along the unique path connecting the vertex with the root.

Theorem: If an edge labeled tree $(\mathcal{G}, c)$ satisfies the properties of the previous theorem, then $(\mathcal{G}, c) = (\mathcal{G}(\pi), c(\pi))$ for some p-adic permutation π

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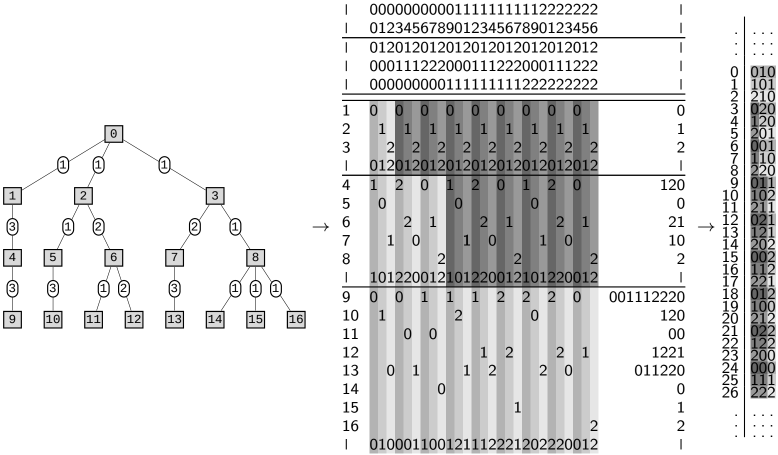

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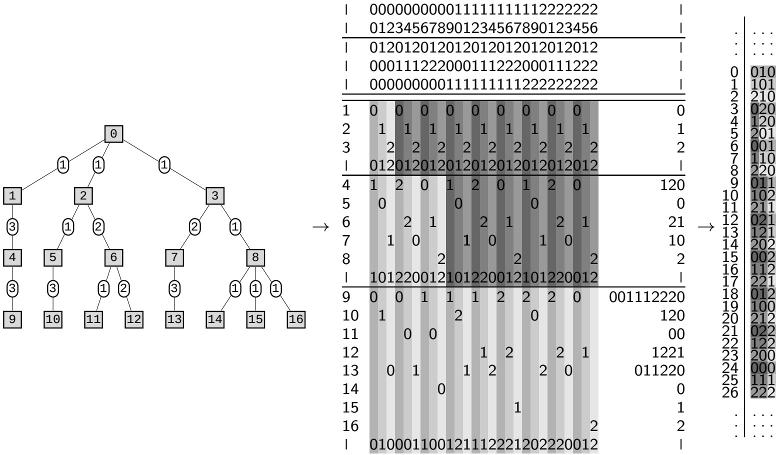

$F(\underline{27}) \,\%\, 9 = (1, 0, 1, 5, 5, 0, 0, 1, 5, 7, 3, 7, 2, 2, 3, 3, 4, 2, 4, 6, 4, 8, 8, 6, 6, 7, 8)$

$(\mathcal{G}, c) = (\mathcal{G}(\pi_{F,(x,x,x)}), c(\pi_{F,(x,x,x)}))$

Theorem:

$$S_{p,k} := \{\text{isom. class of } T \mid F, G\ \mathbb{Z}_p\text{-polynomial } p\text{-adic systems}$$
$$T \text{ full } k\text{-layer rooted subtree of } (\mathcal{G}(\pi_{F,G}), c(\pi_{F,G}))\}$$

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$$|\sigma| > 1 \text{ for root } (\ell, \sigma) \text{ of } T\}$$

$$U_{p,k} := \{\text{isom. class of } T \mid f \in \mathbb{Z}_p[x]\ p\text{-permutation polynomial}$$
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Then,

$\|S_{2,2}\| = 5$	$\|S_{2,3}\| = 20$	$\|S_{2,4}\| = 71$
$\|T_{2,2}\| = 5$	$\|T_{2,3}\| = 12$	$\|T_{2,4}\| = 50$
$\|U_{2,2}\| = 5$	$\|U_{2,3}\| = 18$	$\|U_{2,4}\| = 83$
$\|V_{2,2}\| = 3$	$\|V_{2,3}\| = 5$	$\|V_{2,4}\| = 7$

Sets on next slide

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Corollary: There are p-adic permutation polynomials that cannot be realized by $\mathbb{Z}_p$-polynomial p-adic systems and vice versa

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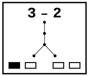
Corollary ("No Y property"):



Corollary: There are p-adic permutation polynomials that cannot be realized by $\mathbb{Z}_p$-polynomial p-adic systems and vice versa

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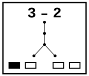
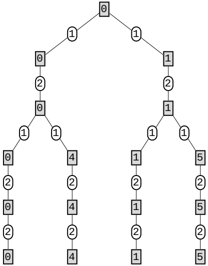


Consequence: $(\mathcal{G}(\pi), c(\pi))$ for $\pi := \pi_{(7x,3x+1),(x-6,5x+9)}$ completely described!



- Prove $D((x^{2000000} + x, x - 1))[n]$ aperiodic for some concrete $n \in \mathbb{Z}$

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Open problems

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Thank you!